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THE ROLE OF BLACK HOLES IN THE ADS/CFT CORRESPONDENCE

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Abstract

The two descriptions of low energy dynamics of coincident branes are considered. One description is the supergravity approximation of the superstring theory where the extremal black p-brane solution emerges. The other description consist of a gauge theory which describes the dynamics of the world volume of the N coincident branes. The AdS/CFT conjecture is introduced and the important equivalence of the two theories' partition functions. By the identification that the UV limit of the field theory corresponds to the boundary of five dimensional anti-de Sitter space, the principle of holography is introduced. To evaluate the partition function the saddle-point approximation is used. However, since the action is divergent due to the infinite volume of asymptotically anti-de Sitter spacetimes, a renormalisation scheme is needed. This is carried out by two different methods. First, the method of holographic renormalisation is used to define a renormalised action and to extract the stress-energy tensor of the field theory on the boundary of the asymptotically anti-de Sitter spacetimes. Secondly, a finite action is obtained by performing a background subtraction. In the latter method, a detailed analysis is performed in order to determine the thermodynamical favourable spacetime configuration as a function of temperature. It is found that a phase transition occurs from thermal anti-de Sitter space to a configuration with a black hole. For the strongly coupled dual field theory in the large N limit, this is interpreted as a phase transition from a confined phase to a deconfined phase.

Contents

1 Introduction

String theory is one of the most promising candidates that theoretical physicists have for a consistent unifying theory of the forces of nature. Surprisingly, it was found that string theory is not only a theory of strings, but, in fact, also of spatially extended objects called branes. There are two low energy descriptions of branes. The supergravity approximation that yields an effective action for which the geometry of branes can be solved for, and a $U(N)$ gauge theory that describes the dynamics of the world volume of the branes. It is from these two descriptions that the gauge/gravity dualities arise. A duality relates a strongly coupled theory to a dual weakly coupled theory.

A particularly interesting gauge theory is QCD, which is based on the gauge group SU(3). QCD is a quantum field theory where the basic constituents of hadrons are quarks and gluons. It is the theory used to describe the strong forces and is an asymptotically free theory which means that the cutoff $\Lambda_{\rm QCD}$ decreases as the energy increases. At low energy it is thus strongly coupled. With a gauge/gravity duality, it becomes possible to approach the intractable strongly coupled regime of the theory by performing perturbation theory at weak coupling in the dual gravity theory. Although the dual string theory for QCD is not yet known, it is believed to exist. Currently, dualities that relate their conformal invariant cousins with a $SU(N)$ gauge group to string theory are considered. These gauge theories are often supersymmetric with a large number of degrees of freedom, that is, large N. Nonetheless, qualitative predictions for more realistic theories, like QCD, can still be made.

In this text, a duality known as a AdS/CFT correspondence is approached in the form where the type IIB superstring theory on the maximally symmetric $AdS_5 \times \mathbf{S}^5$ background is related to the four dimensional supersymmetric $\mathcal{N} = 4$ Yang-Mills gauge theory with gauge group $SU(N)$.

Anti-de Sitter space is the maximally symmetric solution of Einstein's equations with a negative cosmological constant, and the four dimensional gauge theory is a conformal invariant theory; that is, it has a vanishing beta function. Since string theory, and in particular the type IIB theory, plays such a dominant role, appendix B provides a general introduction and gives, among other things, the massless spectrum and the brane spectrum of the type IIB theory. The AdS/CFT correspondence is quite remarkable, as already mentioned, since it relates a theory of gravity, which string theory is, to a theory with no gravity at all. It should, however, be kept in mind at this point that it is only conjectured, not proven.

When matching spacetime coordinates of the two theories, it emerges that the UV limit of the gauge theory corresponds to the conformal boundary structure of anti-de Sitter space. This is the limit where field theories are usually discussed, which is why the field theory is said to live on the boundary of anti-de Sitter space. This picture is called a holographic duality because the higher dimensional physics in anti-de Sitter space is encoded in the lower dimensional field theory. It turns out, however, that the choice of coordinates introduces an ambiguity because it leads to different conformal structures of the boundary. However, this choice is simply a choice of regulator at a given energy scale when renormalising the divergent action of anti-de Sitter space. The negative curved anti-de Sitter space has a number of natural choices of coordinates, for which appendix A is provided.

The correspondence can be generalised to a duality between type IIB on asymptotic anti-de Sitter backgrounds and a dual gauge theory based on considerations of near-extremal solutions of supergravity. The interior of such backgrounds can, among other things, contain black holes. Black holes have a vast history of thermodynamical analogy for which a detailed review is given in appendix D. The analogy means that it is possible to introduce a temperature in the correspondence. This opens for an approach to make predictions in the strongly coupled field theory at finite temperatures. A finite temperature has the implication of introducing an energy scale in the field theory, which breaks its conformal invariance.

Along with the correspondence, the equivalence of the two theories' partition functions are likewise conjectured [29]. From the Euclidean path integral approach towards quantum gravity, which provides a semi-classical prescription for evaluating the thermodynamical partition function of the gravity theory, the stable and favourable spacetime configurations as a function of temperature can be determined. In doing so, one considers the positive-definite sections of the contributing metrics, which are obtained by performing a Wick rotation. In contrast to the anti-de Sitter metric, the black hole metric admits a unique spin structure corresponding to a thermal (anti-periodic) boundary condition, which is supersymmetry-breaking. The relation between the Euclidean path integral and the partition function of the canonical ensemble is reviewed in appendix C. Having the most favourable spacetime configuration as a function of temperature, possible phase transitions are directly predictable and can be mapped to the strongly coupled dual gauge theory at large N. In fact, one finds that the gauge theory in question undergoes a transition from a confined phase consisting of singlet hadrons to a deconfined phase of quark-gluon plasma. Interestingly, this plasma, in the case of QCD, has been observed briefly at the experiments at the Relativistic Heavy Ion Collider (RHIC) [32], [4].

The two low energy descriptions of N coincident branes are introduced in section 2. In particular, the extremal black 3-brane solution of supergravity, which turns out to have the special near-horizon geometry of $\mathrm{AdS}_5 \times \mathbf{S}^5$, and the four dimensional gauge theory living on the the D3-branes, which turns out to have the special property of conformal invariance. Looking at the descriptions in their respective limits of validity at low energies in section 3, the AdS/CFT correspondence considered here is introduced. A discussion of the holographic nature of the duality will also be given. In section 4 it is shown how a finite temperature in the correspondence arises from the black hole geometry of the near-extremal black 3-brane. Furthermore, the divergent action due to the infinite volume of anti-de Sitter spacetime and the need for a renormalisation are discussed. The method of holographic renormalisation is used in section 4.5 to extract the boundary stress-energy tensor of the anti-de Sitter and Schwarzschild-AdS spacetimes. This also spawns an elaborate discussion of the impact the choice of coordinates has on the theory. Hereafter, the $\mathcal{N} = 4$ gauge theory on the spatial manifold $\mathbf{S}^3$ shall be considered at finite temperature. To study this theory at finite temperatures, the thermodynamical partition function is determined on $\mathbf{S}^1 \times \mathbf{S}^3$ by Euclideanising the spacetime metrics and evaluating the saddle-point approximation of the Euclidean path integral. Having the partition function makes it possible to compute the thermodynamical properties of the black hole in anti-de Sitter space and compare it to the empty anti-de Sitter space. Finally, the results are compared to the expected behaviour of the dual field theory.

The units are chosen such that $c = \hbar = k = 1$ while Newton's constant G_{N} is kept.

2 Supergravity and Gauge Theory

As hinted in appendix B, there exist two descriptions of the low energy dynamics of branes. One description treats the brane as a source for the closed string fields in superstring theory and is a solution to the low energy field equations derived from supergravity. The other description uses the dynamics of the world volume field theory of N coincident Dp-branes. In both cases, it is important to identify valid regimes. First, recall that there exists a whole tower of stringy dynamics on top of the low energy physics, whose influence is made insignificant when $\alpha' \to 0$. In addition, working with small g_s ensures that one remains in the classical supergravity regime, that is at the string tree level. Also, in the weak-coupling regime it is evident from the tension of the D-brane that one can treat them as rigid objects and therefore the truncation of the world volume effective actions can be trusted. The two descriptions are essential for the understanding of the conjecture of the duality between gauge theory and gravity.

In the following section, the extremal black p-brane solutions of the type II supergravity actions shall be considered. The case of $p = 3$ turns out to be particular interesting in that it, among other things, has a finite horizon area. The solution arises from the self-dual four-form gauge potential in the type IIB R-R sector and its near-horizon geometry turns out to be $\mathrm{AdS}_5 \times \mathbf{S}^5$. Later, when more is understood, the non-extremal black 3-brane solution will be considered. It will turn out to have a Schwarzschild-anti-de Sitter black hole times a five-sphere geometry. In section 2.2 and 2.3, the open string sectors and coincident D-branes shall be considered. This gives a $\mathrm{U}(N)$ gauge theory, and of particular interest is the world volume field theory of N coincident D3-branes which is the four dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $\mathrm{SU}(N)$ gauge group.

2.1 Extremal Black p-Branes

This section will show how the anti-de Sitter space emerges in the AdS/CFT correspondence. Considering the type II supergravity approximation, solutions are obtained for the weak-coupling $g_s \ll 1$ regime where strings are propagating in a fixed background. The solutions will turn out to be similar to the charged Reissner-Nordström solutions known from general relativity (see section D.2). They are, however, extended in p spatial directions and because they have an event horizon they are called black p-branes. In particular, the focus here will be on the case of imposing the extremal condition on the general p-brane solution. Recall that extremal objects in general relativity are defined by $M = Q$ and as a consequence have zero temperature[1]. This is similar for the higher dimensional p-brane generalisations. The explicit form of a general p-brane solution is not relevant for the discussion, but can be found in various references (see e.g. [17], [1]). Finally, it should be mentioned that although the supergravity solution is the description used to describe low energy string theory, it is believed that the solution may be extended to the full type II string theory. Although, it will of course have α' corrections of the metric and other fields.

As for the black hole solutions in four dimensions, which are solutions to the equations of motion of the Einstein-Hilbert action, one is similarly considering the solutions to the equations of motion derived from the Type II supergravity action. To construct an extremal black p-brane it is only necessary to include one field from the R-R sector in the action, say C_{p+1} (see section B.1). One finds a family of ten dimensional solutions which sources gravity, the dilaton, and the R-R gauge potential. The two-form term from the NS-NS sector vanishes. Let the field strength of the R-R sector field be $F_{p+2} = dC_{p+1}$ and Φ be the dilaton then the string-frame

[1]The extremal condition is given from the definition of the action given by equation (1) and does therefore not include a factor of $\sqrt{G_{\mathrm{N}}}$.

action takes the form [4]

$$S^{(p)} = \frac{1}{2\kappa_0^2}\int d^{10}x\sqrt{-g}\left[e^{-2\Phi}(\mathcal{R}+4(\partial\Phi)^2)-\frac{1}{2}|F_{p+2}|^2\right] \quad , \tag{1}$$

where κ_0 is given by equation (74). The case of $p = 3$ is special because the field strength arising from the R-R four-form of Type IIB is self-dual. To ensure this is satisfied, it is necessary to impose the additional constraint

$$F_5 = \star F_5 \quad ,$$

and introduce an additional factor of 1/2 in front of the $|F_5|^2$ term. The solution for the extremal black p-brane metric is

$$ds^2 = H_p^{-\frac{1}{2}}\left(\eta_{\mu\nu}dx^\mu dx^\nu\right) + H_p^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_{8-p}^2\right) \quad . \tag{2}$$

The part in the first parenthesis is the flat Lorentzian metric in $p+1$ dimensions while the part in the second parenthesis is an Euclidean metric in $9-p$ dimensions. This is consistent with the fact that placing an extremal p-brane in spacetime breaks Lorentz symmetry,

$$SO(9,1) \rightarrow SO(p,1)\times SO(9-p) \quad , \tag{3}$$

where the first factor is the Lorentz symmetry along the brane while the second factor is the rotation symmetry for the perpendicular directions. Since there is rotational symmetry in the $9-p$ transverse directions, it is possible to use spherical coordinates with a radial coordinate r and angular coordinates on a $(8-p)$-sphere.

The function H_p present in the metric is the harmonic function that solves the $(9-p)$-dimensional Laplace equation. Extremal black p-branes are defined by these harmonic functions which are functions of the radial coordinate r,

$$H_p(r) = 1 + \left(\frac{r_p}{r}\right)^{7-p} \quad . \tag{4}$$

The solution is thus parametrized by the positive parameter r_p to be determined later. From the dilaton's equation of motion one finds the solution

$$e^{\Phi} = g_s H_p^{\frac{3-p}{4}} \quad ,$$

from which it is seen that the dilaton field is spatially varying for all values of p except $p = 3$. From the behaviour $H_p \to 1$ as $r \to \infty$ of the harmonic function g_s defines the string coupling at infinity. This behaviour also shows that the metric is asymptotically flat. For $p < 3$, it is evident that the system goes into the nonperturbative regime for $r \to 0$. This means e^{Φ} becomes large for which the classical supergravity solution is unreliable. Interestingly, for the special case of $p = 3$ the dilaton is seen to be constant and thus the string coupling is given by g_s for all r.

One can generalise to a multicentre solution by writing the harmonic function as

$$H_p(r) = 1 + \sum_{i=1}^{k} \left(\frac{r_p}{|\vec{r} - \vec{r}_i|} \right)^{7-p} \quad , \tag{5}$$

which represents k parallel extremal p-branes located at arbitrary positions given by $\vec{r}_i$. For this particular form, each brane carries N_i unit of R-R charge and the total charge is the integer N. In the following, a single brane with N R-R charges is considered. From the equation of motion of the R-R field one finds

$$\begin{aligned} C_{p+1} &= (H_p^{-1} - 1) dx^0 \wedge ... \wedge dx^p \\ F_{p+2} &= dH_p^{-1} \wedge dx^0 \wedge ... \wedge dx^p \quad . \end{aligned}$$

Since the brane will generate a flux of the corresponding $p+2$ field strength, one must be able to express the above equation for the field strength by the following

$$F_{p+2} = \begin{cases} Q(\omega_5 + \star\omega_5) & \text{for } p = 3 \\ Q \star \omega_{8-p} & \text{otherwise} \end{cases} \quad ,$$

where Q is the D-brane charge per unit volume and ω_n is the volume form of a n-sphere. Note that the special case of $p = 3$ is due to the self-dual field strength. This particular form ensures that the integration over the dual field strength $\star F$ yields the charge [1]

$$\int_{S^{8-p}} \star F_{p+2} = Q\Omega_{8-p} = N \quad , \tag{6}$$

where N is the R-R charge of the brane. The volume of a n-sphere is

$$\Omega_n = \mathrm{Vol}(\mathbf{S}^n) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \quad . \tag{7}$$

One can express the parameter r_p in terms of g_s and N from equation (6). Alternatively, it can be done by matching the metric (2) with the form of a Schwarzschild black hole with mass M in ten dimensions. In the Einstein-frame, the time-time component h reads [4]

$$h = 1 - \frac{r_0^2}{r^{d-3}}, \quad r_0^2 = \frac{16\pi G_{\mathrm{N}}^{(d)} M}{(d-2)\Omega_{d-2}} \quad . \tag{8}$$

However, in order to match this result with the corresponding term in the string-frame metric, an appropriate transformation of metric and dilaton must be performed (see e.g. [17]). This reveals that if the H_p is given by equation (4) with $d = 10 - p$, then [23]

$$r_p^{d-3} = \frac{16\pi G_{\mathrm{N}}^{(d)} M}{(d-3)\Omega_{d-2}} \quad . \tag{9}$$

The mass of the extremal p-brane wrapped around a p-dimensional compact space of volume V_p can be expressed in terms of the tension and charge of a Dp-brane as

$$M = N T_{\mathrm{D}p} V_p, \quad T_{\mathrm{D}p} = \frac{1}{(2\pi)^p g_s \ell_s^{p+1}} \quad .$$

At infinity in the reduced $(d-p)$-dimensional spacetime the brane appears as a point source of mass M. Newton's constant in ten dimensions is related to the $(d-p)$-dimensional one by

$$G_{\mathrm{N}}^{(d-p)} = \frac{G_{\mathrm{N}}^{(d)}}{V_p}, \quad G_{\mathrm{N}}^{(10)} = 8\pi^6 g_s^2 \ell_s^8 \quad . \tag{10}$$

From the two last relations, one has

$$G_N^{(d-p)} M = G_N^{(d)} N T_{\mathrm{D}p} \quad ,$$

which, when substituted into equation (9) with $d = 10 - p$, yields a relation between the parameter r_p in units of string length and $g_s N$

$$\left(\frac{r_p}{\ell_s}\right)^{7-p} = (2\sqrt{\pi})^{5-p} \Gamma\left(\frac{7-p}{2}\right) g_s N \quad . \tag{11}$$

For a general p the squared curvature invariant is proportional to either side of this equation. For r_p much larger than the string length ℓ_s, that is $g_s N \gg 1$, the supergravity description can therefore be trusted. Further discussion of this equation in the case of $p = 3$ is addressed later. For all p except for $p = 3$, the horizon at $r = 0$ is a singular place of zero area, since the radius of $\mathbf{S}^{8-p}$ vanishes there. The metric therefore only describes the spacetime outside the horizon. The 3-brane is of interest for various reasons; its world volume has four dimensional Poincaré invariance, it is self-dual, and the horizon has finite area. In addition, the dilaton is constant throughout spacetime, which simplifies the above relation, setting $r_3 = b$ and the string length $\ell_s = \sqrt{\alpha'}$ (see section B.2), one has

$$b^4 = 4\pi g_s N \alpha'^2 \quad . \tag{12}$$

The near-horizon of the 3-brane is also particularly interesting because it describes the low energy physics as shown in section 3.2. In the near-horizon region $r \ll b$, the harmonic function can be approximated by

$$H_3(r) \approx \left(\frac{b}{r}\right)^4 \quad , \tag{13}$$

and thus the geometry of the near-horizon can is found

$$ds^2 \approx \left(\frac{r}{b}\right)^2 (\eta_{\mu\nu} dx^\mu dx^\nu) + \left(\frac{b}{r}\right)^2 \left(dr^2 + d\Omega_5^2\right) \quad . \tag{14}$$

Making a variable substitution $z = b^2/r$ reveals the familiar form of anti-de Sitter spacetime in local coordinates (see equation (68))

$$ds^2 \approx \frac{b^2}{z^2} \left[\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right] + b^2 d\Omega_5^2 \quad . \tag{15}$$

More precisely this is five dimensional anti-de Sitter space times a five-sphere $\mathrm{AdS}_5 \times \mathbf{S}^5$ where the parameter b is identified as the radius of both. The anti-de Sitter part is the primary focus and is reviewed in some details in appendix A. The constant Ricci curvature of AdS is given by the radius of curvature in equation (60)

$$\mathcal{R} \propto -\frac{1}{b^2} \quad . \tag{16}$$

With this equation and constant string coupling g_s it is now possible to compare scales with the coupling constant $g_s N$. Above, the black p-brane solutions have only been treated using classical supergravity. This requires that the curvature is small compared to the string length. By equation (12) and thus equation (16) this implies that b is much larger than the string length. To suppress string loops one also needs to be in the weak-coupling regime $g_s \ll 1$, that is the regime of string perturbation theory. This is possible provided that N is sufficiently large. However, if the string coupling is large, one could also use S-duality $g_s \to 1/g_s$ (see section B.5). In the $p = 3$ case where the horizon is not singular, the solution can, in fact, be analytically extended beyond $r = 0$. The maximally extended metric does not have a singularity and is geodesically complete [1]. The supergravity approximation in the case of $p = 3$ is thus reliable when

$$N > g_s N \gg 1 \quad .$$

Stringy corrections are thus suppressed for $g_s N \gg 1$ and quantum corrections are small when $N \gg 1$ provided $g_s N$ is fixed. The ten dimensional Planck length is related to the string length by $\ell_p^4 = g_s \ell_s^4$, thus equation (12) can be expressed as

$$\left(\frac{b}{\ell_p}\right)^4 = 4\pi N \quad .$$

This means that b must be much larger than the Planck length ℓ_p.

Besides the low energy limit of supergravity $\alpha' \to 0$, the other expansion regime of string theory is that of weak coupling $g_s \to 0$. In this case, equation (5) shows via equation (11) for $g_s N \ll 1$ that the metric for coincident branes essentially becomes flat everywhere except on the $(p+1)$-dimensional hypersurface given by $\vec{r} = 0$, where the metric appears to be singular. Strings propagating on this background are thus moving in flat spacetime, except when the string reaches the brane.

2.2 Non-Abelian Gauge Theory on D-branes

A Dp-brane is a $(p+1)$-dimensional hypersurface in spacetime. They are charged under a $p+1$ gauge field, which is part of the massless closed string modes present in the supergravity multiplet derived for type IIB in section B.1. Note that this spectrum was derived in flat space before any D-branes were present. The branes are therefore sources of closed strings and can have open strings ending on them. The brane spectrum for type IIB is discussed in section B.4. A description of weak-coupling exists and is addressed now.

In the weak-coupling regime where the string coupling g_s is taken to be small, string perturbation theory becomes possible (see section B.2). In this case, a D-brane becomes much heavier than the fundamental string (see section B.4) and is taken to be rigid surfaces where open strings can end, inducing a U(1) gauge theory on the world volume. Branes are thus viewed as point-like in their transverse directions in otherwise flat space. Multiple coincident D-branes allow open strings to start on one brane and end on another by placing Chan-Paton matrices λ^a_{ij} on the ends [4]. For N branes $N^2 - N$ string configurations are possible. In this case, the effective loop expansion parameter for the open strings is $g_s N$ rather than g_s. The D-brane description is therefore valid for $g_s N \ll 1$. If the ends of the open string are labelled i and j, the massless open string state can be shown to be a vector

$$|\mu\rangle \otimes |i\rangle \otimes |j\rangle \quad ,$$

which is a gauge field A^a_μ of the U(N) gauge group. The λ^a_{ij} are generators of the adjoint representation. In the coincident case, the $\mathrm{U}(1)^N$ is thus enhanced to U(N). In the limit $\alpha' \to 0$, only the massless states of the open strings remain and they describe oscillations and the gauge field on the branes. In this limit, the gauge theory is free. When only the dynamics on the branes are of interest, the overall factor $U(1) = U(\mathrm{N})/SU(\mathrm{N})$, which determines the position of the branes, can be ignored. This leaves a SU(N)

gauge symmetry. Superstring theory has 32 supercharges, which form eight four-dimensional spinors; that is, $\mathcal{N} = 8$ in four dimensions. Placing coincident D-branes in spacetime breaks translation invariance and therefore breaks half of the supersymmetry. The fields on the world volume of the coincident D-branes are those of a maximally symmetric vector multiplet for spins less or equal to one: gauge fields, scalars, and spinors, all in the adjoint representation of SU(N). The non-trivial terms in a weak-field expansion are thus exactly $\mathcal{N} = 4$ super Yang-Mills.

2.3 D3-branes; Conformal $\mathcal{N} = 4$ Super Yang-Mills

In this section some essential properties of the SU(N) gauge theory living on the four dimensional world volume of N coincident D3-branes are discussed. For more extensive reviews see [1], [19], or [7].

From the previous section, it is evident that the four dimensional gauge theory living on the coincident D3-branes is the supersymmetric Yang-Mills theory with $\mathcal{N} = 4$ supercharges and gauge group SU(N). In general, it is difficult to find quantum field theories which are conformally invariant. However, this is exactly the case for this particular theory. This comes from the fact that its beta function vanishes everywhere, which implies that there is a cancellation of UV divergences to all orders in perturbation theory. The need for introducing a renormalisation scale is therefore not necessary. One is, however, free to define the theory at a particular energy scale E by integrating out all degrees of freedom above that scale. For the case of super Yang Mills theory in $p + 1$ dimensions the effective coupling can be determined by dimensional analysis

$$g^2_{\text{eff}}(E) \sim g^2_{\text{YM}} N E^{p-3} \quad .$$

For $p = 3$, the coupling is seen to be independent of the energy scale and is known as the 't Hooft coupling $\lambda = g^2_{\text{YM}} N$. It should be mentioned that the open string coupling constant coincides with $g_{\rm YM}$. In the case of $p = 3$, the $\mathcal{N} = 4$ vector multiplet is constituted of the components

$$\left(A^a_\mu, \psi^{ai}, \phi_{[ij]}\right) \quad ,$$

where $i = 1, ..., 4$ is an adjoint $\mathrm{SU}(4)_{\rm R}$ index and $[ij]$ the six-dimensional antisymmetric representation of $\mathrm{SU}(4)_{\rm R}$. The $a = 1, ..., N$ is a $\mathrm{SU}(N)$ gauge group index. Thus, there is one gauge field, four fermions, and $9 - p = 6$ scalars. The label R refers to the global R-symmetry. By definition, this symmetry group does not commute with the supersymmetries. It is evident that the six scalar fields and the four fermions rotate under this group.

It is worth introducing the conformal group. A d-dimensional Lorentizian manifold is conformally flat if the metric can be written [5]

$$ds^2 = e^{u(x)} \left(\eta_{\mu\nu} dx^\mu dx^\nu\right) \quad ,$$

where u is the conformal factor which is allowed to have x dependence. The conformal group is the subgroup of general diffeomorphisms which preserves the conformal flatness of the metric (see section D.6 for a note on diffeomorphisms). The conformal group consists naturally of translations and rotations. In addition, the scale transformation $x^\mu \to \lambda x^\mu$ for some constant λ is also a conformal transformation. Lastly, a conformal transformation, known as the special conformal transformation, constitutes a part of the group. It can be derived in a number of ways (see e.g. [4]). The infinitesimal transformations of the d-dimensional conformal group are then:

- $\delta x^\mu = a^\mu$, translations
- $\delta x^\mu = \omega^\mu{}_\nu x^\nu$, Lorentz transformation
- $\delta x^\mu = \lambda x^\mu$, scale transformation
- $\delta x^\mu = b^\mu (x_\rho x^\rho) - 2x^\mu (b_\rho x^\rho)$, special conformal transformation

a^μ, $\omega^\mu{}_\nu$, λ, and b^μ are taken to be infinitesimal generators of the conformal group. The dimensionality of the group can be counted from these parameters. Recall the Lorentz generators $\omega_{\mu\nu}$ are antisymmetric thus

$$d + \frac{d(d+1)}{2} + 1 + d = \frac{(d+2)(d+1)}{2} \quad .$$

One can derive the Lie algebra by commuting the infinitesimal generators. It turns out to be the non-compact form of the $(d+2)$-dimensional rotation group SO(d,2) in the case of Lorentzian signature. In four dimensions, the conformal group is therefore SO(4,2) with covering group SU(2,2). Finally, it should be mentioned that the conformal group in two spacetime dimensions is special in that it is actually infinitely dimensional.

The large N limit is particularly interesting for the discussion in section 3.1. It turns out that the theory in this 't Hooft limit has a convenient topological expansion of amplitudes. One finds that the contribution of diagrams of genus g scales for large N and fixed coupling λ like N^χ, where χ is the Euler characteristic $\chi = 2 - 2g$. The limit of large N enables expansions in $1/N$. The expansion is seen to be dominated by the surfaces of maximal χ. Thus the leading terms in the expansion consist of surfaces of zero genus; that is, planar diagrams which give a contribution of N^2. The rest of the diagrams are suppressed by factors of $1/N^2$.

3 Anti-de Sitter / Conformal Field Theory

Above, the two low energy descriptions of branes were considered separately. It was seen that their validity was appropriate for different limits of the effective coupling strength $g_s N$. The next sections consider the behaviour of a system consisting of N coincident branes in ten dimensional spacetime when taking the low energy limit of the system in the two opposite limits of $g_s N$. For large coupling one therefore considers a multicentre solution given by equation (5) where each brane effectively carries one unit of R-R charge. The integer N is taken to be large. In both limits, the system is seen to undergo a decoupling into two parts. One part turns out to be equal in both limits and consists of a system of closed strings propagating on flat spacetime. From the low energy limit, conjecturing a duality between string theory and gauge theory comes naturally about.

Some parts of the following analysis are more elaborate in various reviews on the AdS/CFT correspondence (see e.g. [1], [7], [19]). However, the focus here is to understand under which assumptions the low energy limits of near-horizon AdS and gauge theory are appropriate descriptions and why it is possible to have arbitrary finite temperatures in a low energy limit.

3.1 Decoupling in the $g_s N \ll 1$ Limit

Consider N coincident D3-branes in ten dimensional spacetime. Their world volume is a (3+1)-dimensional plane. At zero coupling g_s spacetime is flat Minkowski space as seen from the p-brane metric given by equation (2). On this background, string theory has two kinds of perturbative excitations: open strings on the branes and closed type IIB strings in the spacetime bulk with no interaction taking place. Now, considering a small but non-zero coupling $g_s N \ll 1$, spacetime will still be approximatively flat. At low energies – that is, energies that are much smaller than the string scale $E \ll 1/\ell_s$ or equivalent keeping all energies bounded $E \leq E_0$ while taking the limit $\alpha' \to 0$ – the massive states of the open strings on the D-branes become too heavy to be observed and only the massless string states can be excited (see section B.3). In this limit, one can write down an effective system consisting of three components. Firstly, the massless closed string states of the ten dimensional type IIB supergravity multiplet derived in section B.1 living in the ten dimensional bulk. Secondly, the massless states of the open strings on the (3+1)-dimensional world volume of the branes, which is $\mathcal{N} = 4$ SU(N) super Yang-Mills theory. And thirdly, the interactions between the two; the open string modes and the closed string modes. For example, two colliding open strings on the brane could form a closed string and peel off into the spacetime as Hawking radiation [19]. Both of the two first contributions have higher derivative corrections in powers of α' to their effective Lagrangians.

To understand how the effective description behaves in the low energy limit, one can consider the strength of the interactions. The strength of the interactions between the closed strings in the bulk, but also between the spacetime fields and the fields on the branes, are determined by the ten-dimensional Newton constant $G_{\rm N} \sim g_s^2 \alpha'^4$. Keeping the energy, g_s, N, as well as all other dimensionless parameters fixed while taking $\alpha' \to 0$, the two types of interactions both vanish along with the higher derivative corrections and leave the two systems completely decoupled.

3.2 Decoupling in the $g_s N \gg 1$ Limit

The D3-brane geometry presented in section 2.1 is a valid description of superstring theory provided $g_s N \gg 1$. Recall for the $p = 3$ case, the dilaton is constant and if necessary can be made small by a S-duality transformation. The solution is not singular and have a finite horizon located at $r = 0$. The metric is given by equation (2) with the harmonic function given by (4). For this static D3-brane solution of supergravity, the energy of an object

E_r at a fixed position r measured by an outside observer at infinity will be redshifted when the objects come closer and closer to the horizon. For a note on redshift, see the discussion above equation (87). From the timelike Killing vector field ξ^μ, which is well-defined everywhere for AdS, the redshift factor $V = \sqrt{-\xi^\mu \xi_\mu}$ is seen to be a function of r,

$$E_\infty = V E_r = H_3^{-\frac{1}{4}} E_r \quad .$$

It is thus apparent from the behaviour of the harmonic function H_3 that in the near-horizon limit $r \ll b$ where the harmonic function can be approximated by equation (13), the redshifted energy

$$E_\infty \approx \frac{r}{b} E_r \approx \frac{r}{\sqrt{\alpha'}} E_r \tag{17}$$

shows that the near-horizon describes low energy physics.

At low energies, two excitations are possible. One, due to the redshift finite excitations that either emanate from the horizon or are brought close to the horizon, and another far from the branes' low energy excitations consisting of massless particles propagating in spacetime will have large wavelengths. In the low energy limit, these two excitations decouple and do not interact. The near horizon excitations can not overcome the gravitational potential $\sim g_{00}$ and escape to infinity while the horizon, which is small compared to the wavelength of the massless particles, has negligible interaction cross-section. The result is two non-interacting systems: a system of low energy closed strings propagating on a flat background and a system of closed type IIB superstrings on the near-horizon geometry.

It is in order to be a little more elaborate, for the purpose of later discussion, on how it is possible to consider arbitrary excited states near the horizon. Energies in the near-horizon region $r \ll b$ in string units are kept fixed $\sqrt{\alpha'} E_r$ while $\alpha' \to 0$ such that the energy as measured from infinity is given by equation (17), thus to keep the energy fixed while taking $r \to 0$ implies that $u = r/\alpha'$ should be kept fixed.

3.3 Conjecture

The above analysis considered a system of N D3-branes in ten dimensional spacetime in two opposite limits of the effective coupling g_sN. It was shown that at low energies both limits of g_sN had a part consisting of closed strings propagating in a flat background. It was shown that these closed strings did not interact with the other part of the system. The closed strings can be identified to be the same in both limits. In fact, it must be the same for all values of g_s. It is then natural to identify the other part of the system in the two limits to be the same also. Since the two remaining theories are completely different theories and neither can be treated non-perturbatively it is difficult to prove they are in fact the same. Therefore, one conjectures that the $\mathcal{N} = 4$ SU(N) super Yang-Mills theory in 3+1 dimensions is a dual theory to the full quantum type IIB superstring theory on $\text{AdS}_5 \times \mathbf{S}^5$. This is the strongest form of the conjecture, however there exist three forms of the conjecture with various strengths (see [7]).

In the above analysis, the weakest form was considered, which was what gave rise to the conjecture originally. It is that the gravity description is valid for large g_sN, but the full string theory might not agree with the field theory. A slightly stronger conjecture is to say the two theories are the same for finite g_sN, but only when N is large. This is α' corrections would agree, but not necessarily g_s corrections. The strongest form of the conjecture states that the two theories are exactly the same for all values of g_s and N. Later, when the role of the non-extremal 3-brane is considered a more general statement of the AdS/CFT correspondence emergence where the four dimensional gauge theory is related to spacetimes which are only asymptotically $\text{AdS}_5 \times \mathbf{S}^5$. The interior of spacetime is thus allowed to contain all kinds of processes like excited fundamental string states or black holes, etc. [1]. The field theory is thus an effectively sum over all spacetimes which are asymptotic to $\text{AdS}_5 \times \mathbf{S}^5$.

A duality requires a precise map between quantities in the two related theories. The different mappings between quantities are said to constitute a dictionary for the correspondence. As a start, the relation between the dimensionless parameters of the two theories is considered. First, the rank of the SU(N) gauge group, which is $N-1$, is present in supergravity by equation (6) as the five-form flux through the five-sphere. This came about by enclosing the D3-branes, which carry a total of N units of D3-brane charge. Secondly, the coupling constant in the Yang-Mills g_{YM} and in the string coupling g_s have an exact relation realised by the S-duality that both theories have (see section B.5). Setting the two scalars τ equal in the two theories reveals the relation

$$g_{\mathrm{YM}}^2 = 4\pi g_s \quad .$$

For the case of large N the effective coupling constant for the gauge theory is the 't Hooft coupling constant $\lambda = g_{\mathrm{YM}}^2 N$. Using above relation between the couplings, one can write

$$\lambda = g_{\mathrm{YM}}^2 N = 4\pi g_s N \quad . \tag{18}$$

Furthermore, from the case of extremal black D3-branes one has the radius of curvature of the AdS_5 and five-sphere in units of the string length given by equation (12). Using equation (18) this relation can be expressed in terms of the 't Hooft coupling constant

$$\left(\frac{w}{\ell_s}\right)^4 = 4\pi g_s N = \lambda \quad . \tag{19}$$

Now that a mapping between the parameters is made, one can express the five dimensional Newton's constant in the parameters of the field theory. From equation (7) and (10), the ten dimensional Newton's constant can be related to the five dimensional

$$G_{\mathrm{N}}^{(5)} = \frac{G_{\mathrm{N}}^{(10)}}{\mathrm{Vol}(S^5) w^5} = \frac{8\pi^6 g_s^2 l_s^8}{\pi^3 w^5} \quad ,$$

since the volume of a unit five-sphere is π^3. Using the mapping from the correspondence given by equation (19)

$$g_s^2 \ell_s^8 = \frac{w^8}{16\pi^2 N^2} \quad ,$$

one obtains,

$$G_{\rm N}^{(5)} = \frac{8\pi^6 w^8}{16\pi^5 N^2 w^5} = \frac{\pi w^3}{2N^2} \quad .$$

The five dimensional constant is of interest, since the correspondence includes the five dimensional anti-de Sitter spacetime. Results calculated in the following sections can therefore be converted into field theory quantities using the derived expression for the five dimensional Newton's constant.

It is worth mentioning that ensuring a match between the symmetries of the two theories provides a suitable consistency check for the duality. The near-horizon geometry given by equation (15) has an isometry from the AdS_5 part, which is SO(4,2) and a SO(6) rotation symmetry of the five-sphere. Correspondingly, section 2.3 showed that the conformal group in four dimensions was SO(4,2) and additionally had a $SU(4) \sim SO(6)$ R-symmetry. Usually, the cover groups are used since the fermions belong to the spinor representations.

3.4 Holographic Duality

Taking a point x^μ in AdS to correspond to a position in the field theory the subtle question arise of how the radial coordinate should be interpreted in the gauge theory and what $z = 0$ corresponds to. Recall that equation (17) showed that the radial coordinate controlled the energy as measured from infinity. It turns out that z in fact represents an energy scale E of the gauge theory. To see this, recall that for a conformal theory one can perform a scale transformation in the gauge theory.

$$x^\mu \to \lambda x^\mu \quad \Rightarrow \quad E \to E/a \quad .$$

Since a point in the gauge theory is taken to be a point in AdS, this transformation should also be possible on the metric. Applying the transformation on the AdS metric implies the rescaling $z \to z/\lambda$ and reveals exactly the redshift relation obtained before.

The limit $z \to 0$ defines the boundary of AdS_5. Any radial slice is conformal to Minkowski space in four dimensions as is seen from equation (15). The boundary corresponds then to the field theory where no degrees of freedom have been integrated out ($E \to \infty$). This is the limit of UV in the gauge theory. Since this is the limit that is normally discussed, one often says that the field theory lives on the boundary. However, the field theory really lives everywhere. A slice of z corresponds to a particular effective theory at that cutoff. This is the notion of holography: the physics on the five dimensional AdS is encoded in the four dimensional gauge theory. It should be mentioned that the boundary is in the IR limit of the AdS theory while it is UV for the gauge theory. Quite generally, there exists a holographic relation where physics on the $(d+1)$-dimensional anti-de Sitter space can be encoded in the dual d-dimensions conformal field theory. Note that the five-sphere was ignored in the above discussion as it will be for a large part of the following.

The location of the field theory on the boundary clarifies the former statement that the field theory effectively is a sum over all spacetimes which are asymptotically AdS. Although until now only the vacuum AdS has been introduced, another contributing spacetime shall be derived from the consideration of the non-extremal black p-brane solution in section 4.2. Dual gauge theories in such cases are, however, not conformally invariant.

Now that the UV of the gauge theory has been connected to the boundary of asymptotically AdS space, an important statement of the correspondence can be made. The duality claims that there should be a bulk field ϕ for every gauge invariant operator $\mathcal{O}_\phi$ [25]. One makes the identification that the bulk partition function equals the generating functional of the field theory correlation functions [29]

$$Z_{\text{AdS}}(\phi_{0,i}) = Z_{\text{CFT}}(\phi_{0,i}) \quad . \tag{20}$$

Here the quantities $\phi_{0,i}$ on the gravity side should be interpreted as the boundary values of the fields $\phi_i|_{z=0}$ propagating in asymptotic AdS. Thus, the fields $\phi_{0,i}$ are functions of the four coordinates describing the boundary of AdS_5. On the field theory side, the $\phi_{0,i}$ correspond to external sources or currents that are coupled to operators. Much of the dictionary comes from equating the two partition functions of the two theories which is why this identification can be interpreted as the actual duality.

4 Finite Temperature

The correspondence between gauge theory and superstring theory is now postulated and it should be possible to make predictions in one theory by calculations in the corresponding theory. The primary interest will be to consider the theories at finite temperature. The approach taken here is to investigate the finite temperature physics of asymptotically AdS spacetimes and look at the corresponding behaviour in the related gauge theory. In section 4.2, a contribution to the partition function shall be derived from the consideration of the decoupling limit of the non-extremal p-brane solution. It turns out to be a Schwarzschild black hole solution in AdS_5 times a five sphere. Like black holes in asymptotically flat space, black holes in asymptotically AdS also have thermodynamical properties. A discussion of black holes in flat space is given in appendix D. The Schwarzschild black hole solution in AdS will turn out to be a competing contribution in the thermodynamical partition function. Global coordinates are introduced in section 4.3.

In section 5, the contributing spacetimes shall be Euclideanised and the thermodynamical analogy of black holes shall be utilised. A brief note on the relation between statistical partition function and the path integral approach towards quantum gravity is given in appendix C. It will be shown that a phase transition occurs between the thermal AdS space and the AdS black hole solution as a function of temperature. With help from the AdS/CFT duality, the result can be interpreted in the gauge field as a transition from a confined phase to a deconfined phase. There is, however, another approach to computing thermodynamical quantities. In section 4.5, the results of applying holographic renormalisation on asymptotic AdS spaces shall be discussed. In particular, the extraction of the boundary stress-energy tensor.

4.1 The Saddle-Point Approximation

In order to do calculations, the AdS partition function can be evaluated using the saddle-point approximation to the path integral

$$\exp\left(-I_{\mathrm{AdS}}(\phi_0)\right) = \left\langle \exp\left(\int \phi_0 \mathcal{O}\right)\right\rangle_{\mathrm{CFT}} \quad , \tag{21}$$

where I_{AdS} is the classical Euclideanised supergravity action and $\mathcal{O}$ is the dual field theory operators. Actually, both sides are not well-defined without renormalisation. On the right hand side, the correlation functions have UV divergences. Similarly, the left hand side is divergent due to the infinite volume of the non-compact AdS spacetime. In the following, the left hand side and its divergences will be focused on.

The action of some $(n+1)$-dimensional spacetime is given by the Einstein-Hilbert action plus the Hawking-Gibbons surface term [9]

$$I_{\mathrm{bulk}} + I_{\mathrm{surf}} = -\frac{1}{16\pi G_{\mathrm{N}}}\left(\int_{\mathcal{M}} d^{n+1}x\sqrt{g}(R-2\Lambda) + \int_{\partial\mathcal{M}} d^n x\sqrt{\gamma}\, 2K\right) \quad . \tag{22}$$

Here $K = K^i{}_i$ is the trace of the extrinsic curvature of the boundary $\partial\mathcal{M}$ and γ_{ij} the induced metric on the boundary. Recall that the extrinsic curvature measures how curved a hypersurface embedded in spacetime is. In the case of a spherical symmetric spacetime with radial coordinate r, the computation of the extrinsic curvature of a spherical hypersurface is simplified significantly. Here, the embedding function reads $f = r - R = 0$ for some constant R. This is of interest, since the boundary can be obtained by taking R to infinity. The outpointing unit vector normal to the spherical surface is given by

$$n^\mu = \frac{1}{\sqrt{|g_{rr}|}}\left(\frac{\partial}{\partial r}\right)^\mu \quad . \tag{23}$$

The embedding coordinates of the surface in the spherical symmetric spacetime are coincident with the spacetime coordinates therefore the extrinsic curvature can be expressed

$$K_{ij} = \frac{1}{2} n^\mu \frac{\partial \gamma_{ij}}{\partial x^\mu} \quad . \tag{24}$$

The trace is readily given by

$$K = \gamma^{ij} K_{ij} = \frac{1}{2}\gamma^{ij}\left(n^{\mu}\frac{\partial \gamma_{ij}}{\partial x^{\mu}}\right) \quad . \tag{25}$$

Now, as mentioned previously, the classical AdS action suffers from being divergent. In fact, both terms of the action are divergent. In section 5.5, the action of an asymptotic AdS spacetime is shown to be proportional to the volume of the spacetime, but since this volume is infinite the Einstein-Hilbert action is divergent. Furthermore, in section A.6 it is shown that the metric diverges on the boundary and that it is in fact not possible to induce a metric there. Thus, to obtain finite quantities, like the total energy, some approach to renormalisation must be taken. Two different approaches shall be addressed: holographic renormalisation in section 4.5 and background subtraction in section 5.5.

4.2 Near-extremal D3-brane

From the holographic principle it was evident that the UV limit of the field theory lived on the boundary of AdS. Spaces that are asymptotic anti-de Sitter is therefore of interest since they contribute to the partition function. As shall be shown in this section, one such space arises from the horizon of the non-extremal solutions of supergravity. Like non-extremal black hole solutions, black p-brane solutions have nonzero temperature. This will play an essential role for the discussion of section 5. Restricting attention to $p = 3$, the metric of the non-extremal black 3-brane is given by (see e.g. [17])

$$ds^2 = H_3^{-\frac{1}{2}}\left[-Z dt^2 + \delta_{ij} dy^i dy^j\right] + H_3^{\frac{1}{2}}\left[Z^{-1} dr^2 + r^2 d\Omega_5^2\right] \quad ,$$

where $i, j = 1, 2, 3$. H_3 is given by equation (4) and Z is a function of the radial coordinate

$$H_3(r) = 1 + \left(\frac{b}{r}\right)^4 , \quad Z(r) = 1 - \left(\frac{r_H}{r}\right)^4 \quad .$$

The black brane horizon is located at $r = r_H$. In the extremal case, it was found that the parameter $r_p = r_3 = b$ was related by equation (11) to the combination $g_s N$. This, although not stated, is in fact true for the general p-brane solution [17]. Thus, using this relation the essential operation is to take the decoupling limit where the near-horizon decouples from the bulk $g_s N \gg 1$. This limit of the non-extremal case corresponds to a near-extremal solution and the harmonic function can be approximated by (13) again

$$\left(\frac{b}{\ell_s}\right)^4 \propto g_s N \gg 1 \quad \Rightarrow \quad H_3 \approx \left(\frac{b}{r}\right)^4 \quad .$$

One could expect that the limit would have restricted attention to the region near the horizon and both removed the singularity and the asymptotically flat region. However, this is not the case as is evident from the form of the metric

$$ds^2 = \frac{r^2}{b^2}\left[-Zdt^2 + \delta_{ij}dy^i dy^j\right] + \frac{b^2}{r^2}Z^{-1}dr^2 + b^2 d\Omega_5^2 \quad .$$

The first part is the Schwarzschild-AdS$_5$ black hole metric in local coordinates (Poincaré coordinates) where the horizon is $\mathbb{R}^3$ instead of S^3 while the second part is the five-sphere as in the extremal case. The metric is asymptotic to the anti-de Sitter space and has an inverse temperature $\beta = \pi b^2/r_H$. As mentioned in appendix section A.1, the Schwarzschild-AdS metric is, like the AdS metric, a solution to Einstein's equations with a negative cosmological constant. In order to compare the form here to equation (15), a change of variables $z = b^2/r$ is performed

$$ds^2 = \frac{b^2}{z^2}\left[-Zdt^2 + \delta_{ij}dy^i dy^j + Z^{-1}dz^2\right] + b^2 d\Omega_5^2 \quad . \tag{26}$$

In these coordinates the horizon is located at $z_0 = b^2/r_H$ and the temperature is $\beta = \pi z_0$. The thermodynamical quantities temperature, energy, and entropy will be given a proper treatment in section 5.

Now, two contributing spacetimes to the partition function have been identified. The string theory partition function evaluated by the saddle-point approximation given by equation (21) is thus

$$Z_{\text{string}} \simeq e^{-I_{\text{AdS}}} + e^{-I_{\text{BH}}} \quad .$$

Here, the I_{BH} denotes the action of the Schwarzschild-AdS$_5$ black hole. Section 5 will work with this approximation in detail. However, some questions in relation to classical black holes in flat space are worth mentioning. Most of the classical work done on black holes assumes asymptotic flatness, stationarity, and other various things reviewed in section D.5. The theorems developed therefore rely upon such assumptions. Addressing black holes in asymptotic anti-de Sitter space naturally rises questions about theorems of uniqueness, positive mass, initial values, etc. (some of these questions are addressed in [20] and [31]) In addition, section D contains a complete macroscopic description of the thermodynamic analogy of four dimensional black holes in asymptotic flat space. The results of black hole thermodynamics are here directly extended to the discussion of their higher dimensional generalisations.

4.3 Global Coordinates

In sections 2.1 and 4.2, the two geometries of the black Dp-branes have been derived in Poincaré coordinates given by equation (15) and (26), respectively. These coordinates only cover a part of the manifold, but one could also consider global covering coordinates. Appendix A contains among other things a discussion of different coordinate choices for anti-de Sitter space. In particular, mappings between them are presented. In this section, the two metrics given in global coordinates by equation (59) are generalised to $d = n + 1$ spacetime dimensions, that is

$$ds^2 = -V dt^2 + V^{-1} dr^2 + r^2 d\Omega_{n-1}^2 \quad , \tag{27}$$

where $d\Omega_{n-1}^2$ is the metric of a $(n-1)$-dimensional unit sphere $\mathbf{S}^{n-1}$. The covering space of anti-de Sitter space is given with the static form

$$V(r) = 1 + \frac{r^2}{b^2} \quad , \tag{28}$$

where the radius of curvature is denoted by b. The period of the associated imaginary time coordinate is arbitrary for the metric stated. However, a constraint is later required in order to relate it to the Schwarzschild black hole solution which can be introduced in anti-de Sitter space by

$$\begin{aligned} V(r) &= 1 + \frac{r^2}{b^2} - \frac{r_0^2}{r^{n-2}} = 1 + \frac{r^2}{b^2} - \frac{w_n M}{r^{n-2}} \\ w_n &= \frac{16\pi G_{\mathrm{N}}}{(n-1)\mathrm{Vol}(\mathbf{S}^{n-1})} \quad , \end{aligned} \tag{29}$$

where r_0 was the term encountered for the Schwarzschild black hole in flat space in equation (8) and is here used to define w_n such that M is the mass of the black hole as shown later. G_{N} denote the $(n+1)$-dimensional Newton's constant and $\mathrm{Vol}(\mathbf{S}^{n-1})$ the $(n-1)$-spherical surface area of the corresponding unit sphere. Note that the function V given by (29) tends asymptotically towards that of anti-de Sitter space given by (28) as r tends to infinity.

4.4 The Conformal Boundary

Section A.6 shows that AdS evaluated on the boundary has a second order singularity. For this reason it is necessary to pick a positive function f, called the defining function, such that

$$g_{(0)} = f^2 g|_{\partial\mathcal{M}}$$

constitutes an induced metric on the boundary. For equation (71), which is the metric for a hypersurface of constant radial coordinate of the metric by equation (29), a natural choice could be $f = b/r$, for example. This procedure, which can be done for any asymptotic AdS space, defines a conformal structure. The metric $g_{(0)}$ depends, however, on the choice of f and different choices are related by a conformal transformation $f' = fe^u$.

A natural metric for dual field theory in the UV is obtained by removing the conformal factor r^2/b^2 making the above choice of f. This gives a conformal structure in global coordinates of $\mathbb{R} \times \mathbf{S}^{n-1}$. Correspondingly, the choice $f = z/w$ for the defining function in local coordinates given by equation (14) leads to the conformal boundary $\mathbb{R} \times \mathbb{R}^{n-1}$. Thus, in some sense the physics looks like it depends on the choice of coordinates, but by the choice of coordinates the definition of the Hamiltonian will also be affected and the physics is, as expected, in fact the same. Since the radial coordinate plays the role of an energy scale in the gauge theory, the choice of coordinates can also be seen as a choice of regulator at that energy scale [17]. This shall be evident from the treatment of holographic renormalisation in section 4.5.

One can show that the two boundaries of local and global coordinates are conformally related in their Euclideanised versions. Removing the conformal factor in equation (15) using the above choice of the defining function, the boundary takes the following form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad .$$

Now considering the Euclideanised version of the metric $\eta_{\mu\nu} \to \delta_{\mu\nu}$ and writing the Euclideanised space in polar form

$$\delta_{\mu\nu} dx^\mu dx^\nu = dx^2 + r^2 d\Omega_{n-1}^2 \quad .$$

Introducing the coordinate $x = \ln \tau$ one has $dx^2 = r^2 d\tau^2$, thus

$$dx^2 + r^2 d\Omega_3^2 = x^2 (d\tau^2 + d\Omega_3^2) \quad .$$

This is exactly the form of the boundary in the global coordinates given by equation (69) times the conformal factor x^2. This can be seen by setting the coordinate τ equal to it.

4.5 Holographic Renormalisation

In this text, the primary method applied in detail when determining thermodynamical quantities related to asymptotic AdS spaces will be background subtraction, following Witten [30]. However, it is worth considering a different approach known as holographic renormalisation. This is a quite far-reaching approach and has a natural interpretation in the dual field theory. The essential key points needed to obtain the stress-energy tensor will be addressed in this section. For more information, see [25], [6], [2].

From the discussion of section 3.3 and 3.4 and the findings of section 4.2, asymptotic AdS spacetime solutions to Einstein's equations with a negative cosmological constant are of natural interest. Given some representative of the conformal structure as defined in section 4.4, it can be shown that one can obtain a solution of Einstein's equations which is asymptotic AdS [25]. The metric in the neighbourhood of the boundary $z \to 0$ takes the form

$$\begin{aligned} ds^2 &= \frac{b^2}{z^2}\left(dz^2 + g_{ij}(x,z)dx^i dx^j\right) \\ g_{ij}(x,z) &= g_{(0)ij} + z g_{(1)ij} + z^2 g_{(2)ij} + \ldots \quad . \end{aligned}$$

Einstein's equations can be solved order by order in z to determine the coefficients $g_{(k)ij}$ for $k > 0$. All the coefficients can be determined uniquely in terms of $g_{(0)}$. It turns out that all coefficients in odd powers of z vanish up to order z^n. Notice that choosing $g_{(0)ij} = \delta_{ij}$ with all other coefficients zero reduces the solution to the AdS metric given in Poincaré coordinates by equation (15). To simplify computation, one can introduce the coordinate $\rho = z^2$

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = b^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho)dx^i dx^j\right) \\ g_{ij}(x,\rho) &= g_{(0)} + \ldots + \rho^{\frac{d}{2}} g_{(d)} + h_{(d)}\rho^{\frac{d}{2}}\log\rho + \mathcal{O}(\rho^{\frac{d+1}{2}}) \quad . \end{aligned} \tag{30}$$

Any asymptotic AdS metric can be brought into this form near the boundary, which is located at $\rho = 0$. The logarithmic coefficient $h_{(d)}$ is only non-zero when n is even. If the bulk metric is conformally flat, an exact solution to Einstein's equations exists and is given by [24]

$$\begin{aligned} g_{ij}(x,\rho) &= g_{(0)} + g_{(2)}\rho + g_{(4)}\rho^2 \\ g_{(2)ij} &= \frac{1}{n-2}\left(\mathcal{R}_{ij} - \frac{\mathcal{R}}{2(n-1)} g_{(0)ij}\right) \\ g_{(4)ij} &= \frac{1}{4}(g_{(2)})^2_{ij} = \frac{1}{4} g_{(2)ip}\, g^{pq}_{(0)}\, g_{(2)qj} \quad , \end{aligned} \tag{31}$$

where the curvatures $\mathcal{R}$ and $\mathcal{R}_{ij}$ are given by the metric $g_{(0)}$.

Now that an expansion for the solution of an asymptotic AdS spacetime is given, the corresponding action given by equation (22) can be addressed. It is evident that both integrals are divergent; the bulk action is proportional to the infinite volume of spacetime while the surface term diverges since the induced boundary diverges at the boundary. To render the action finite, one must perform a renormalisation scheme. First step is to introduce a regularisation procedure. This is done by introducing a regulator (or cutoff) ϵ on the radial coordinate such that the bulk action is evaluated for $\rho \geq \epsilon$ (recall that the boundary is at $\rho = 0$) and the surface term at $\rho = \epsilon$. Using the asymptotic solution given by (30), one can evaluate the regularised action as a power series of ϵ [24]. Restricting to $n = 4$, the regularised action is

$$I_{\text{reg}} = \frac{1}{16\pi G_{\text{N}}} \int d^n x \sqrt{g_{(0)}} \left(\epsilon^{-2} a_{(0)} + \epsilon^{-1} a_{(2)} - \log \epsilon\, a_{(4)}\right) + \mathcal{O}(\epsilon^0) \quad . \tag{32}$$

The regularisation scale ϵ can be thought of as specifying a spatial hypersurface a finite amount within the interior of spacetime. Interestingly, in addition to the terms addressed below, one finds a logarithmic divergence. For $n = 4$ it takes the value [15]

$$a_{(4)} \log \epsilon = b^3 \left(\frac{1}{8}\mathcal{R}^{ij}\mathcal{R}_{ij} - \frac{\mathcal{R}^2}{24}\right) \log \epsilon \quad .$$

Note that this vanishes for a flat background. To make the action finite, one can perform a minimal subtraction; that is, to subtract the pieces that divergence as one goes to the boundary by the corresponding counter terms. These terms are a finite set of boundary integrals, which only depend, like the coefficients given in equation (30), on the induced conformal structure at the boundary $g_{(0)}$. Including the counter terms, the renormalised action is written

$$I_{\rm ren}[g_{(0)}] = \lim_{\epsilon \to 0} \left(I_{\rm reg}[g_{(0)}, \epsilon] + I_{\rm ct}[g_{(0)}, \epsilon] \right) \quad .$$

Only two counter terms and the additional logarithmic one are needed to cancel all divergences for the case of $n = 4$. They take the form [17]

$$I_{\rm ct}[g_{(0)}, \epsilon] = \frac{1}{16\pi G_N} \int_{\rho=\epsilon} d^n x \sqrt{-\gamma} \left[\frac{6}{b} + \frac{b}{2}\mathcal{R} - a_{(4)} \log \epsilon \right] \quad . \tag{33}$$

The metric γ_{ij} is the metric on the boundary induced by restricting ρ to be the constant ϵ. Note that the same metric was also used in the surface term to compute the form given by equation (32). The $\mathcal{R}$ and $\mathcal{R}_{ij}$ are the Ricci scalar and Ricci tensor for the metric γ_{ij}, respectively.

From this point of view, it is now possible to understand the discussion given in section 4.4 of why the choice of coordinates (or, more precisely, the choice of conformal boundary structure) seems to affect the physics. The counter terms are given by the coefficients $g_{(2)}, ..., g_{(n-2)}$, which are uniquely determined on the choice of $g_{(0)}$. $g_{(0)}$ is, however, only determined up to a conformal transformation, and since the logarithmic divergence is regularisation independent the action depends on the choice of conformal boundary structure $g_{(0)}$ [15]

$$I[e^u g_{(0)}] = I[g_{(0)}] + \mathcal{A}[g_{(0)}, u] \quad .$$

This anomalous transformation is known as the holographic Weyl anomaly and is only present for odd spacetime dimensions. Although the anomaly may vanish in one background, its metric variation may not. The stress-energy tensor can therefore have a contribution from the anomalous variation, which is the case for AdS in global coordinates. In the following, the logarithmic term will be neglected.

The stress-energy tensor at the boundary can be extracted from the renormalised action in the following way

$$T_{ij} = \frac{2}{\sqrt{-g_{(0)}}} \frac{\delta I_{\text{ren}}}{\delta g^{ij}_{(0)}} \quad .$$

One can evaluate this expression by first computing the stress-energy tensor at a finite distance within the interior (that is in the regulated theory) and thereafter go to the boundary (removing the regulator)

$$T_{ij} = \lim_{\epsilon \to 0} \frac{2}{\sqrt{-g(x,\epsilon)}} \frac{\delta I_{\text{ren}}}{\delta g^{ij}(x,\epsilon)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{\frac{n}{2}-1}} \frac{1}{\sqrt{-\gamma}} \frac{\delta I_{\text{ren}}}{\delta \gamma^{ij}} \quad .$$

This way, one can express the stress-energy tensor in terms of the induced metric γ_{ij}

$$T_{ij}[\gamma] = T^{\text{reg}}[\gamma] + T^{\text{ct}}[\gamma], \quad T^{\text{reg}}[\gamma] = \frac{1}{8\pi G_N} \left(K_{ij} - K\gamma_{ij} \right) \quad ,$$

where the regulated contribution comes from the regulated action and K_{ij} and K is the extrinsic curvature and its trace as defined in equation (24) and (25). The counter term contribution is calculated from the boundary integrals given by equation (33). The second counter term is proportional to an Einstein-Hilbert action and thus the finite stress-energy tensor in the case of $n = 4$ is [2]

$$T_{ij}[\gamma] = \frac{1}{8\pi G_N} \left(K_{ij} - K\gamma_{ij} + \frac{3}{b}\gamma_{ij} + \frac{b}{2} \left[\mathcal{R}_{ij} - \frac{1}{2}\mathcal{R}\gamma_{ij} \right] \right) \quad . \tag{34}$$

Using this formula the stress-energy tensor can be determined for the AdS_5 metric given in local coordinates by equation (14). The outpoint normal vector is given by equation (23)

$$n^{\mu} = \frac{r}{b} \left(\frac{\partial}{\partial r} \right)^{\mu} \quad .$$

The extrinsic curvature of the boundary and its trace is

$$K_{00} = -\frac{r^2}{b^3}, \quad K_{11} = K_{22} = K_{33} = \frac{r^2}{b^3}, \quad K = \gamma^{ij} K_{ij} = \frac{2}{b} \quad .$$

Evaluating the stress-energy tensor by equation (34) it is thus found to vanish. In fact the first counter term is enough to cancel all divergences. A zero stress tensor is to be expected for an empty space, but as shall be shown this depends on the choice of conformal boundary structure.

It has been shown in [24] that one can express the stress-energy tensor in terms of the expansion coefficients of g_{ij} given by equation (30) as

$$T_{ij} = \frac{nb}{16\pi G_N}\left(g_{(d)ij} + X^{(d)}_{ij}\right) \quad , \tag{35}$$

where $X^{(d)}$ depends on the number of dimensions. For odd d, this contribution vanishes. For the case of $n = 4$, which is related to asymptotic AdS_5 spacetimes, one has [6]

$$\begin{aligned} X^{(4)}_{ij} &= -\frac{1}{8}g_{(0)ij}\left[(\mathrm{Tr}\ g_{(2)})^2 - \mathrm{Tr}\ g^2_{(2)}\right] - \frac{1}{2}(g^2_{(2)})_{ij} + \frac{1}{4}g_{(2)ij}\mathrm{Tr}\ g_{(2)} \\ (g_{(2)})^2_{ij} &= g_{(2)ip}\, g^{pq}_{(0)}\, g_{(2)qj} \\ \mathrm{Tr}\ g_{(2)} &= g^{ij}_{(0)} g_{(2)ij} \\ \mathrm{Tr}\ g^2_{(2)} &= g^{ij}_{(0)}\, g_{(2)ip}\, g^{pq}_{(0)}\, g_{(2)qj} \quad . \end{aligned} \tag{36}$$

Using this prescription the stress-energy tensor of the boundary of the conformally flat AdS_5 can be determined. The metric is given in global coordinates by equation (27). It can be brought to the form given by equation (31) by the following coordinate transformation (also used in section A.3)

$$\frac{r^2}{b^2} = \frac{1}{\rho}\left(1 - \frac{\rho}{4}\right)^2 \quad \text{and} \quad t^2 = b^2(x^0)^2 \quad .$$

Choosing the metric $g_{\mathbf{S}^3} = \mathrm{diag}(1, \sin^2\psi, \sin^2\psi\sin^2\phi)$ for the unit three-sphere, the metric coefficients $g_{(0)}, g_{(2)}$ and $g_{(4)}$ can be read off to be

$$\begin{aligned} g_{(0)ij} &= \mathrm{diag}(-1, 1, \sin^2\psi, \sin^2\psi\sin^2\phi) \\ g_{(2)ij} &= -\frac{1}{2}\mathrm{diag}(1, 1, \sin^2\psi, \sin^2\psi\sin^2\phi) \\ g_{(4)ij} &= \frac{1}{16}g_{(0)ij} \quad . \end{aligned} \tag{37}$$

Using equation (36), one finds $X^{(4)}_{ij} = \frac{1}{4}\delta_{i,0}\delta_{j,0}$. Scaling the coefficients by $1/b^2$ to get the stress-energy tensor in ordinary units, this result together with (35) yields

$$T_{ij} = \frac{1}{64\pi G_N b}\left(4\delta_{i,0}\delta_{j,0} + g_{(0)ij}\right) \quad .$$

For the case of global coordinates where the boundary structure is $\mathbb{R} \times \mathbf{S}^3$, the field theory is living on $\mathbf{S}^3$. To obtain the total energy, one integrates over the three-sphere, with volume $2\pi^2 b^3$, thus obtaining

$$E = \frac{3\pi b^2}{32 G^{(5)}_{\mathrm{N}}} \quad .$$

Although a non-zero energy for a vacuum solution might seem odd in relation to what was found for the choice of local coordinates, it is expected from the previous discussion and matches the Casimir energy of the free field theory limit [17], [2]. As was shown in section 4.3, it is possible by performing a conformal transformation to bring the choice of $g_{(0)}$ on a conformally flat form. This is the case treated above in Poincaré coordinates where $g_{(0)}$ is flat. The stress-energy tensor vanishes and the total energy of the field theory on $\mathbb{R}^3$ is zero. The energy thus depends on the choice of $g_{(0)}$ and can be understood as a constant shift in energy.

For the black hole solution given by equation (29) the same method can be applied. Since the metric is asymptotic AdS, it is possible to put it on the form given by equation (30) performing the following coordinate transformation

$$\frac{r^2}{b^2} = \frac{1}{\rho}\left(1 - \frac{\rho}{4}\right)^2 + \frac{r_0^2}{4b^2\rho} \quad \text{and} \quad t^2 = b^2 (x^0)^2 \quad .$$

Expanding in terms of ρ to second order reveals that $g_{(0)}$ and $g_{(2)}$ is given by equation (37) while $g_{(4)}$ takes the form

$$g_{(4)ij} = \frac{1}{16}\mathrm{diag}\left(\frac{12r_0^2 - b^2}{b^2}, (b^2 + 4r_0^2) g_{\mathbf{S}^3}\right) \quad .$$

Again, one finds $X^{(4)}_{ij} = \frac{1}{4}\delta_{i,0}\delta_{j,0}$, but the form of $g_{(4)ij}$ yields an additional term to the total energy of the black hole metric

$$E = \frac{3\pi b^2}{32G_{\rm N}^{(5)}} + \frac{3\pi r_0^2}{8G_{\rm N}^{(5)}} \quad .$$

The first term is exactly what was obtained above for AdS_5 and the energy correctly reduces in the case of $r_0 = 0$. Letting $r_0^2 = w_4 M$ and using the expression (39), the second term is seen to be in agreement with the result obtained using the method of background subtraction given by equation (49). In [2] and [17] the stress-energy tensor has been determined using equation (34).

5 Thermal Phase Transition in Anti-de Sitter Space

As mentioned in section 4.1, the saddle-point approximation will be used to evaluate the path integral. In appendix C.1, the connection between the Euclidean path integral formulation and the canonical statistical ensemble is established. It is shown that a partition function can be formed by an integral over metrics; that is, the assumption that the dominant contribution to the path integral comes from the background fields that extremize the action. The action introduced in equation (22) is therefore of interest.

In this section, the connection is utilised to study the thermodynamical properties of the asymptotic anti-de Sitter backgrounds. Referring to equation (79), the two contributing metrics are the anti-de Sitter space metric and the Schwarzschild-anti-de Sitter metric, respectively. The Euclideanised versions of these metrics are non-singular positive-definite solutions satisfying the required periodic thermal boundary conditions [14], [30]. It will be possible to answer questions of whether one can have a quantum field theory at a finite temperature by considering the thermodynamical stability of the black hole solution. This is readily given by the specific heat. However, the black hole configuration must also be favourable over pure thermal radiation in anti-de Sitter space; that is, have dominant negative free energy. The implication on the dual field theory living on the boundary of the anti-de Sitter space will be discussed.

In section 4.3, global coordinates of the two contributing metrics were introduced and these shall be used exclusively in the following. Throughout the section, the general $(n+1)$-dimensional case will be considered, where n is the number of spatial dimensions. Specific examples are, however, dedicated to the cases of $n = 3$ and $n = 4$, which are of particular interest. The former was considered by Hawking and Page [14] and the latter is the case considered in the near-horizon and decoupling limit of the D3-brane geometry.

5.1 Criterion for Confinement/Deconfinement

As mentioned above and from the discussion of the holographic principle, the string theory on the asymptotic AdS_5 backgrounds is related to a strongly coupled conformal field theory living on the boundary $\mathbb{R} \times \mathbf{S}^3$. In the case of the Schwarzschild-AdS solution, this corresponds to having a dual field theory at a finite temperature. However, a temperature introduces a energy scale in the theory which therefore breaks conformal symmetry. When using the Euclidean approach to quantum gravity the metrics are Euclideanised and the boundary structure changes accordingly to $\mathbf{S}^1 \times \mathbf{S}^3$. This is thus the appropriate structure to consider a field theory on $\mathbf{S}^3$ with finite temperature where supersymmetry-breaking boundary conditions has been imposed in the $\mathbf{S}^1$ direction. In the following, attention is restricted to the $\mathcal{N} = 4$ theory on this finite volume manifold.

A criterion for whether the field theory in question is in a confining phase or a de-confining phase is of interest. This will quantify whether a phase transition occurs in the strongly coupled theory or not at large N. A criterion in finite volume for confinement is whether the free energy is of order $\mathcal{O}(1)$ or of order N^2. When the theory is confining, the order of $\mathcal{O}(1)$ shows that the contribution is the color singlet hadrons, and when it is deconfining, the order of N^2 shows that the contribution is the gauge fields, i.e. the gluons. The $\mathcal{N} = 4$ theory on $\mathbf{S}^1 \times \mathbf{S}^3$ in the large N limit is expected to have a low temperature phase with a free energy of order $\mathcal{O}(1)$ and a high temperature phase with a free energy of order N^2 (see [29]).

5.2 Schwarzschild Black Hole in Anti-de Sitter Space

To determine the temperature of the black hole, one requires that the metric does not exhibit a conical singularity at the horizon. In section 5.3, this is done in detail, but requires an expanding around the horizon. It must therefore be ensured that a positive root actually exists before the temperature can be determined. The horizons are given by the roots of the gravitational potential $V(r)$ given in equation (29)

$$1 + \frac{r^2}{b^2} - \frac{w_n M}{r^{n-2}} = 0 \quad . \tag{38}$$

The largest positive root r_+ of this equation is the black hole event horizon. It is evident from Decartes' sign rule that there is at most one positive root. This, if it exists, must therefore be r_+. This quantity determines the euclidean section, since the positive-definiteness of the metric is only maintained by restricting the radial coordinate to the region $r \geq r_+$. In addition, the largest root also defines the mass M of the black hole

$$M = \frac{r_+^{n-2}}{w_n} \left(1 + \frac{r_+^2}{b^2} \right) \quad , \tag{39}$$

where the constant w_n, given by equation (29), is chosen such that M is in fact the mass. This will also be evident from the computation of the energy given in section 5.6. In the following, the roots for the case of $n = 3$ and $n = 4$ are considered.

The Schwarzschild-anti-de Sitter metric in four dimensions has a classical interest and is presented in some details in appendix A by equation (59). It is, of course, the same as in equation (27) with $n = 3$. For this particular case, $w_3 = 2G_\mathrm{N}$ and $\mathrm{Vol}(\mathbf{S}^2) = 4\pi$, therefore the function V takes the form

$$V(r) = 1 + \frac{r^2}{b^2} - \frac{2MG_\mathrm{N}}{r} \quad . \tag{40}$$

The event horizon is found for a suitable $r = r_+$ satisfying the equation $V(r_+) = 0$. Hence, a cubic equation is given

$$r_+^3 + b^2 r_+ - 2MG_\mathrm{N} b^2 = 0 \quad ,$$

where the real coefficients are given by $a_1 = 0$, $a_2 = b^2$ and $a_3 = -2MG_\mathrm{N} b^2$.

The discriminant D of a cubic equation is given as

$$\begin{aligned} D &= Q^3 + R^2 \\ Q &= \frac{3a_2 - a_1^2}{9} = \frac{a_2}{3} > 0 \\ R &= \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54} = -\frac{27a_3}{54} > 0 \quad . \end{aligned}$$

Hence $D > 0$ which means the cubic equation has two imaginary roots and one real root, say r_+. The real root is given by

$$\begin{aligned} r_+ &= S + T - \frac{1}{3}a_1 = S + T \\ S &= \left(R + \sqrt{D}\right)^{\frac{1}{3}} > 0 \\ T &= \left(R - \sqrt{D}\right)^{\frac{1}{3}} \quad . \end{aligned}$$

This can be seen since $S - T$ is always real leaving the two other roots imaginary. However, the real root should be positive in order to make sense physically. The requirement $S + T > 0$ is equal to R being positive. Hence, there always exists one event horizon in the $n = 3$ Schwarzschild-anti-de Sitter space given by a real positive root.

In the case of five dimensional anti-de Sitter space one has $\mathrm{Vol}(\mathbf{S}^3) = 2\pi^2$ and $w_4 = 8G_\mathrm{N}/3\pi$. The function V takes the form

$$V(r) = 1 + \frac{r^2}{b^2} - \frac{8G_\mathrm{N}}{3\pi}\frac{M}{r^2} \quad .$$

Using $r_0 = w_4 M$ and solving for the roots of the quartic equation $V(r_+) = 0$

$$r_+^4 + b^2 r_+^2 - r_0 b^2 = 0 \quad .$$

For this particular form, one can solve for r_+^2

$$r_+^2 = \frac{b^2}{2}\left(\pm\sqrt{1 + \frac{4r_0^2}{b^2}} - 1\right) \quad .$$

The argument of the square root is greater than one. The term in the parenthesis is therefore positive for the positive sign and negative for the negative sign. Solving for r_+, one therefore finds that the equation has a conjugate imaginary pair of roots and two real roots of equal magnitude, but opposite sign. The positive real root is given by

$$r_+ = \sqrt{\frac{b^2}{2}\left(\sqrt{1+\frac{4r_0^2}{b^2}}-1\right)} \quad .$$

Hence, there exists only one event horizon in the case of $n = 4$ for the Schwarzschild-anti-de Sitter space.

5.3 The Periodicity of Imaginary Time

With the existence of an event horizon it is now possible to turn to determining the temperature. Unlike some spaces, AdS has no natural temperature associated with it. One must impose a periodicity in the imaginary time to construct thermal states, say β. The inverse temperature is thereby determined by the periodicity of imaginary time $\beta = T^{-1}$. Rotating the time coordinate into the imaginary plane by $\tau = it$, one can identify the section for which the metric is positive-definite. This section is given by the constraint $r \geq r_+$ on the radial coordinate, where r_+ is the largest root of equation (38). For the case of $n = 3$ and $n = 4$ it was shown that one positive real root always exists.

The geometry of an Euclideanised Schwarzschild-AdS black hole solution has a conical part. To see this, consider the geometry of a cone which is readily described in terms of a flat space metric in polar coordinates (ρ, θ)

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 \tag{41}$$

with the angular coordinate having a domain of $\theta \in [0, 2\pi - \Delta]$ such that a piece of the plane is missing for $\Delta \neq 0$. For a non-zero choice of Δ, the origin of the plane $\rho = 0$ is singular. Such a conical singularity is not desired if the metric is to describe a black hole where the horizon is non-singular. One must require that the horizon is non-singular even though there can not be a continuation inside the horizon. Hence, the periodicity of the imaginary time is constrained by $\Delta = 0$.

To identify the correct periodicity of the Schwarzschild-AdS solution, consider the expansion of the coefficient $V(r)$ around the horizon $r = r_+$ to first order

$$\begin{aligned} V(\tilde{r}) &\approx \left[\frac{2r_+ + w_n M(n-2)b^2 r_+^{1-n}}{b^2}\right]\tilde{r} \\ &= \left[\frac{nr_+^2 + b^2(n-2)}{b^2 r_+}\right]\tilde{r} \quad , \end{aligned}$$

where the horizon is shifted to zero by defining the coordinate $\tilde{r} = r - r_+$. Note that the zeroth order vanishes due to the defining property $V(r_+) = 0$. The second equality is obtained by using the mass of the black hole in terms of the horizon given by equation (39). Now, defining $\rho = \sqrt{\tilde{r}}$, the metric shows a conical form

$$ds^2 \approx 4V^{-1}\left(d\rho^2 + \frac{\rho^2}{4}\left[\frac{nr_+^2 + b^2(n-2)}{b^2 r_+}\right]^2 d\tau^2\right) + r_+^2 d\Omega_{n-1}^2 \quad .$$

Denoting the period of τ by β, the condition for a smooth non-singular space gives

$$\frac{1}{2}\left[\frac{nr_+^2 + b^2(n-2)}{b^2 r_+}\right]\beta = 2\pi \quad .$$

Hence, the periodicity of the imaginary time is obtained as a function of the black hole radius r_+

$$\beta = \frac{4\pi b^2 r_+}{nr_+^2 + b^2(n-2)} \quad , \tag{42}$$

for which the temperature of the the black hole is given $T = \beta^{-1}$. It is interesting to see how the temperature behaves in different limits of r_+. For a small black hole $r_+ \ll b$

$$\beta \simeq \frac{4\pi r_+}{n-2} \quad ,$$

the temperature is large and for a black hole with large radius $r_+ \gg b$

$$\beta \simeq \frac{4\pi b^2}{nr_+} \quad , \tag{43}$$

the limit shows that the temperature scales with the black hole's size.

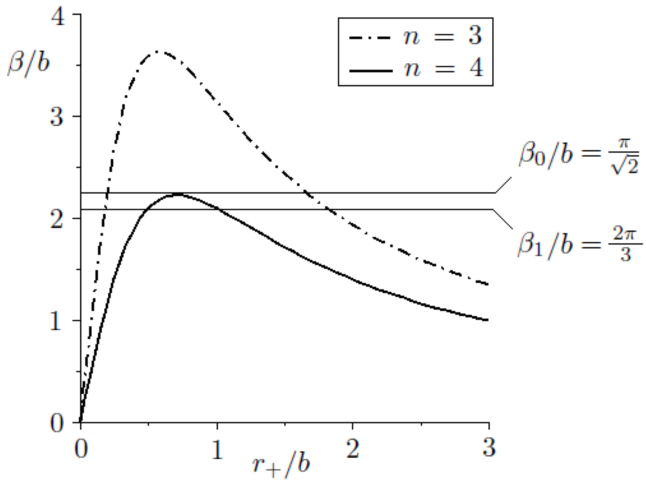

Figure 1: The inverse temperature as a function of the black hole radius depictured for $n = 3$ and $n = 4$. More precisely β/b as function of r_+/b is shown. One can see that at a certain temperature β_0 the two branches of black hole radii evolves. The branching point β_0 are shown for $n = 4$ together with the temperature β_1 considered in section 5.7.

5.4 Two Black Hole Sizes

At a given temperature the allowed sizes of a black hole are given by the periodicity of the imaginary time coordinate τ. The size (horizon radius) r_+ in terms of the inverse temperature β is given by the solving equation (42). Writing

$$n\beta r_+^2 - 4\pi b^2 r_+ + \beta(n-2)b^2 = 0 \quad ,$$

the solutions are

$$r_+ = \frac{2\pi b^2 \pm b\sqrt{4\pi^2 b^2 - n(n-2)\beta^2}}{n\beta} \quad . \tag{44}$$

The real roots are present when the constraint

$$\beta \leq \frac{2\pi b}{\sqrt{n(n-2)}}, \quad n > 2$$

is satisfied. For $n > 2$, equality in the above condition gives the maximum value of $\beta = \beta_0$, which corresponds to a radius $r_0 = b\sqrt{\frac{n-2}{n}}$ of the black hole. This shows that there exists a minimum value of the temperature $T_0 = \beta_0^{-1}$, which initiate two possible black hole radii. At finite temperature above T_0 there are always two valid solutions and the positive branch always has greater radius than the negative branch. The radius of the black hole is therefore a multivalued function of the inverse temperature. The two branches are illustrated for $n = 3$ and $n = 4$ in figure (1). For $n = 2$, there is always two real roots: zero and $2\pi b^2/\beta$ while for $n < 2$ there are no positive real roots when the temperature is real and positive.

5.5 Actions and Partition Functions

Other interesting thermodynamical quantities can be obtained by considering the Euclideanised action. The two metrics given by equation (28) and (29) both extremize the Euclidean action. Keeping the normalisation of the cosmological constant introduced in equation (60), the action given by equation (22) of the maximally symmetric spaces are seen to be proportional to the volume of spacetime

$$\begin{aligned} I &= -\frac{1}{16\pi G_{\rm N}} \int d^{n+1}x\sqrt{g}\,[\mathcal{R} - 2\Lambda] \\ &= -\frac{1}{16\pi G_{\rm N}} \int d^{n+1}x\sqrt{g}\left[-\frac{n(n+1)}{b^2} + \frac{n(n-1)}{b^2}\right] \\ &= \frac{n}{8\pi G_{\rm N} b^2} \int d^{n+1}x\sqrt{g} \quad . \end{aligned}$$

The action has additional surface terms, but for the Schwarzschild-AdS they vanish because the black hole correction in equation (29) vanishes too quickly when approaching the boundary. Since both the anti-de Sitter space and the black hole spacetime have infinite volume, one considers the finite difference between the two actions. The black hole is thus compared to the empty AdS space. This operation can be performed by introducing a regularisation cutoff R for the radial coordinate r

$$\begin{aligned} V_{\text{AdS}} &= \int_0^{\beta'} d\tau \int_0^R \int_{\mathbf{S}^{n-1}} r^{n-1} d\Omega = \frac{\beta' \text{Vol}(\mathbf{S}^{n-1})}{n} R^n \\ V_{\text{BH}} &= \int_0^{\beta} d\tau \int_{r_+}^R \int_{\mathbf{S}^{n-1}} r^{n-1} d\Omega = \frac{\beta \text{Vol}(\mathbf{S}^{n-1})}{n} \left[R^n - r_+^n\right] \quad . \end{aligned}$$

However, in order to compare the two volume integrals, one must make sure they describe the same AdS space asymptotically $r \to \infty$. The black hole solution has a fixed period β given by equation (42) while the anti-de Sitter solution can have an arbitrary period β'. To make a comparison between the two volumes, one must therefore require that the period β' is adjusted such that the geometry of the two spaces equals at the cutoff hypersurface $r = R$. Considering the two metrics at fixed $r = R$ given by equation (27) with V defined by (28) and (29), respectively, one sees that only the time component introduces a difference in the volume since the sphere part is identical. This implies that the two periods β and β' are related by

$$\begin{aligned} \lim_{r=R} \int d^{n+1}x \sqrt{g_{\text{BH}}} &= \lim_{r=R} \int d^{n+1}x \sqrt{g_{\text{AdS}}} \\ \int_0^{\beta} d\tau \sqrt{1 + \frac{R^2}{b^2} - \frac{w_n M}{R^{n-2}}} &= \int_0^{\beta'} d\tau \sqrt{1 + \frac{R^2}{b^2}} \\ \beta \sqrt{1 + \frac{R^2}{b^2} - \frac{w_n M}{R^{n-2}}} &= \beta' \sqrt{1 + \frac{R^2}{b^2}} \quad , \end{aligned}$$

where M is a function of r_+ given by equation (39). The period β' can be expressed in terms of β and approximated in the limit of the cutoff R tending to infinity

$$\begin{aligned} \beta' &= \beta \sqrt{1 - \frac{w_n M b^2}{R^n + b^2 R^{n-2}}} \\ &\approx \beta \left[1 - \frac{1}{2} \frac{w_n M b^2}{R^n}\right] = \beta \left[1 - \frac{1}{2} \left(1 + \frac{r_+^2}{b^2}\right) \frac{r_+^{n-2} b^2}{R^n}\right] \quad . \end{aligned}$$

Note that for large R, the expansion of the square root is valid. Using the large R limit of β' the action difference becomes

$$\begin{aligned} I &= \frac{n}{8\pi G_{\rm N} b^2} \lim_{R\to\infty} \left(V_{\rm BH}(R) - V_{\rm AdS}(R)\right) \\ &= \frac{\beta \mathrm{Vol}(\mathbf{S}^{n-1})}{8\pi G_{\rm N} b^2} \left(\left[R^n - r_+^n\right] - \left[R^n - \frac{1}{2}\left(r_+^{n-2}b^2 + r_+^n\right)\right]\right) \\ &= \frac{\beta \mathrm{Vol}(\mathbf{S}^{n-1})}{16\pi G_{\rm N} b^2} \left(r_+^{n-2}b^2 - r_+^n\right) \quad . \end{aligned}$$

Inserting the inverse temperature β given by (42), the action difference is

$$I = \frac{\mathrm{Vol}(\mathbf{S}^{n-1})}{4G_{\rm N}} \left[\frac{b^2 r_+^{n-1} - r_+^{n+1}}{n r_+^2 + (n-2)b^2}\right] \quad . \tag{45}$$

The partition function of the black hole solution is

$$-I = \log Z_{\rm BH} - \log Z_{\rm AdS} = \log \frac{Z_{\rm BH}}{Z_{\rm AdS}} \quad ,$$

and the free energy

$$\begin{aligned} F_{\rm BH} - F_{\rm AdS} &= -\frac{1}{\beta}\left[\log Z_{\rm BH} - \log Z_{\rm AdS}\right] \\ &= \frac{\mathrm{Vol}(\mathbf{S}^{n-1})}{16\pi b^2 G_{\rm N}} \left[b^2 r_+^{n-2} - r_+^n\right] \quad . \end{aligned} \tag{46}$$

A phase transition becomes possible if the free energy goes from positive to negative; that is, a transition from empty space to a black hole. This is exactly what happens. For sufficiently small r_+ the free energy is positive and for sufficiently large r_+ it is negative for all values of n. However, for this black hole configuration to be stable the heat capacity will have to be positive. The heat capacity is considered in the next section.

In relation to section 5.4 where it was found that at a given temperature two possible radii of the black hole could occur, it is evident from the free energy that the large radius always will be more favourable than the small radius. In fact, the small radius always has positive free energy, which means that it is less probable than empty space, as will be shown in the next section.

5.6 Energy, Specific Heat Capacity

From the temperature it was possible to match the period of AdS space to that of the Schwarzschild-AdS black hole at infinity and thus determine the action difference. Given the action difference and thereby the partition function, the free energy was given. It is possible to determine other thermodynamical quantities as well: the average energy, the specific heat, and the entropy. The energy allows a consistency check of the mass M defined by equation (39) and the specific heat capacity to be calculated. The heat capacity will answer the question about stability of the two black hole configurations. The entropy shall be considered in section 5.8.

From appendix C.1, the average energy is given by

$$\langle E \rangle = \frac{\partial I}{\partial \beta} = \frac{\partial I}{\partial r}\frac{\partial r}{\partial \beta} = \frac{\partial I}{\partial r}\left(\frac{\partial \beta}{\partial r}\right)^{-1} \quad .$$

The action is given by equation (45) in terms of the location of the horizon r_+, which is a multivalued function of the inverse temperature. There are two points to note about the above expression of the energy. First, since β is a function of r_+, the energy should also be a function of r_+. One should not be confused that there are two consistent radii of black holes at a given temperature. Second, expressing the energy as above, mathematically speaking, the energy is not well-defined at the branching point r_0 of the two radii solutions, since the derivative of the inverse temperature will be zero. However, physically, the energy is expected to be defined and finite at r_0.

To perform the calculation, it is helpful to start by using the form of the denominator to substitute the function β in

$$I = \frac{\mathrm{Vol}(\mathbf{S}^{n-1})}{4G_\mathrm{N}}\frac{\beta}{4\pi b^2}\left[b^2 r_+^{n-2} - r_+^n\right] \quad .$$

Defining $\gamma = \mathrm{Vol}(\mathbf{S}^{n-1})/(16\pi G_\mathrm{N} b^2)$, the derivative is

$$\frac{\partial I}{\partial \beta} = \gamma\left(b^2 r_+^{n-2} - r_+^n\right) + \gamma\left((n-2)b^2 r_+^{n-3} - n r_+^{n-1}\right)\beta\frac{\partial r_+}{\partial \beta} \quad . \tag{47}$$

Making the arbitrary choice of focusing on the branch of $r_+ > r_0$, which is given by choosing the positive sign in equation (44), the partial derivative with respect to β is

$$\frac{\partial r_+}{\partial \beta} = -\frac{(n-2)b}{\sqrt{D}} - \frac{2\pi b^2 + b\sqrt{D}}{\beta^2 n} \quad ,$$

where $D = 4\pi^2 b^2 - \beta^2 (n-2)n$ is the determinant. Noting that the second part of the above expression contains r_+/β the following combination can be written as

$$\beta \frac{\partial r_+}{\partial \beta} = r_+ \left[\frac{n r_+^2 + (n-2)b^2}{(n-2)b^2 - n r_+^2} \right] \quad . \tag{48}$$

Substituting this

$$\begin{aligned} \frac{\partial I}{\partial \beta} &= \gamma \left(b^2 r_+^{n-2} - r_+^n \right) + \gamma r_+^{n-2} \left(n r_+^2 + (n-2) b^2 \right) \\ &= \gamma (n-1) \left[r_+^n + b^2 r_+^{n-2} \right] \quad . \end{aligned}$$

An expression for the energy in terms of the horizon is thus obtained

$$\langle E \rangle = \frac{(n-1)\mathrm{Vol}(\mathbf{S}^{n-1})}{16\pi G_\mathrm{N}} \left[b^{-2} r_+^n + r_+^{n-2} \right] = M \quad . \tag{49}$$

As indicated, this corresponds exactly to the definition of the black hole mass given in equation (39) with the choice of w_n. The energy is seen as expected to be positive for $r \geq 0$ for all n, which also means that it is finite at r_0.

It is now possible to compute the specific heat and address the question about the stability of the black hole configuration

$$C_\mathrm{V} = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} = -\gamma (n-1) \left[n r_+^{n-1} + b^2 (n-2) r_+^{n-3} \right] \beta^2 \frac{\partial r_+}{\partial \beta} \quad .$$

Using the result from equation (48)

$$\beta^2 \frac{\partial r_+}{\partial \beta} = \frac{4\pi b^2 r_+^2}{(n-2)b^2 - n r_+^2} \quad .$$

The specific heat capacity is obtained in terms of the horizon radius

$$C_\mathrm{V} = \frac{(n-1)\mathrm{Vol}(\mathbf{S}^{n-1})}{4 G_\mathrm{N}} \left[\frac{n r_+^{n+1} + b^2 (n-2) r_+^{n-1}}{n r_+^2 - (n-2) b^2} \right] \quad . \tag{50}$$

When the dominator is zero, the function goes from negative to positive. At this point, which was already encountered in section 5.4, namely $r_0 = b\sqrt{\frac{n-2}{n}}$, a bifurcation happens. The result shows that the branch for which the black hole radius r_+ is greater than r_0 always has positive specific heat. While, the branch for which the radius r_+ is lesser than r_0 always has negative specific heat. The larger radius is therefore always a thermodynamical stable configuration, but need not always be the most favourable.

Since the branch of black hole radii larger than r_0 is always a stable configuration it is of interest to see if there is a temperature for such radii at which the free energy becomes negative. At such a temperature, the black hole configuration will be more probable than pure thermal radiation in an AdS background. By restricting attention to the solution given by equation (44) with positive sign, the radius can be considered a function of β. Substituting $r_+(\beta)$ into the free energy given by equation (46) one finds that the free energy is zero at

$$\beta_1 = \pm\frac{2\pi b}{n-1} \quad . \tag{51}$$

In section 5.4, it was shown that an event horizon existed for $n \geq 2$. Assuming this and discarding the negative solution of β_1, the derivative of the free energy evaluated at β_1 is

$$\left.\frac{\partial F}{\partial \beta}\right|_{\beta=\beta_1} = \frac{\mathrm{Vol}(\mathbf{S}^{n-1})(n-1)^2 b^{n-3}}{16\pi^2 G_{\mathrm{N}}}, \quad n \geq 2 \quad , \tag{52}$$

which is seen to be positive. The free energy therefore changes sign from positive to negative for decreasing values of β. Thus, the black hole will be energetically favourable when the temperature is greater than $T_1 = \beta_1^{-1}$. The corresponding radius of the black hole is independent of n and is $r_1 = b$.

When $n > 2$, one can check for consistency that $\beta_1 \leq \beta_0$ or equivalent $T_0 \leq T_1$; that is, the black hole solution becomes stable and favourable at a higher temperature than the branching temperature T_0. One obtains the relation

$$\frac{n^2 - 2n}{n^2 - 2n + 1} \leq 1, \quad n > 2 \quad ,$$

which is always true, although the difference $|\beta_1 - \beta_0|$ becomes smaller and smaller as the spatial dimensions n increases.

5.7 Phase Transition

This section summaries the findings of the previous sections about the phase transition as a function of temperature. It should be mentioned that the cases of $n = 3$ and $n = 4$ are completely similar to the general case. Table (1) shows specific values of the aforementioned specific points of events.

n	β_0	r_0	β_1	r_1
3	$\frac{2\pi b}{\sqrt{3}}$	$\frac{b}{\sqrt{3}}$	πb	b
4	$\frac{\pi b}{\sqrt{2}}$	$\frac{b}{\sqrt{2}}$	$\frac{2\pi b}{3}$	b
$\vdots$				
n	$\frac{2\pi b}{\sqrt{n(n-2)}}$	$b\sqrt{\frac{n-2}{n}}$	$\frac{2\pi b}{n-1}$	b

Table 1: Particular values of the bifurcation temperature β_0 and phase transition temperature β_1 for the cases of $n = 3$, $n = 4$, and the general case. The temperature is given by $T = \beta^{-1}$. Note that $r_1 > r_0$ and $\beta_1 < \beta_0$.

For temperatures lower than T_0, the only equilibrium which is possible is thermal radiation in anti-de Sitter space. When the temperature reaches T_0, there are, in addition to the pure thermal radiation, two possible black holes. Only the larger of them has positive specific heat as was shown from equation (50). Using equation (44), the free energy given by (46) can be expressed in terms of β for the small radius and the large radius, respectively. For the larger black hole, the free energy has a positive root at β_1 given by equation (51). Looking at the derivative of the free energy at the point β_1 given by equation (52), one finds that the free energy goes from positive to negative with decreasing β.

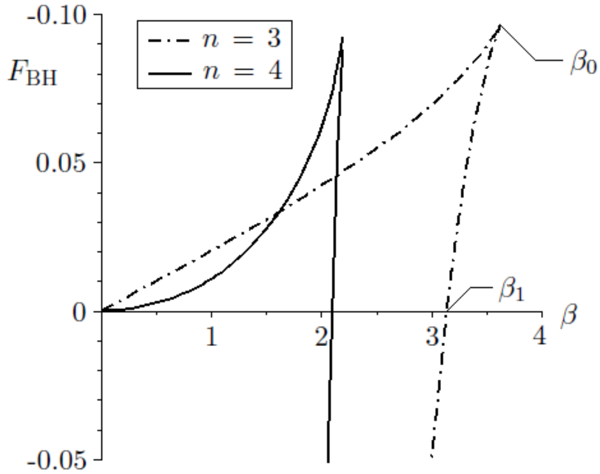

Figure 2: The free energy based on the partition function of the black hole solution as a function of inverse temperature. The radius of curvature is fixed to $b = 1$ and Newton's constant to $G_{\rm N}^{(4)} = G_{\rm N}^{(5)} = 1$. The bifurcation point β_0 is indicated on one of the curves and shows the initiation of two branches of possible black hole radii. For the larger black hole radius where $r_+ > r_0$ a negative free energy is obtained at temperatures $\beta < \beta_1$. The other branch of small mass black holes shows a positive free energy for all values of β.

The free energy of the two black hole radii as a function of temperature is illustrated in figure (2) for the specific cases of $n = 3$ and $n = 4$. It is evident that for the lower value of r_+, the free energy is positive in the entire range as expected, while for the greater value of r_+ the free energy gets negative at the point β_1. At the temperature $T \gtrsim T_1$, the black hole will have lower free energy than pure thermal radiation. The configuration will therefore be more probable and is at least locally stable as seen from the specific heat, equation (50). When the temperature exceeds some point $T_2 > T_1$, the thermal radiation will inevitably collapse to the higher mass black hole.

5.8 Entropy and Field Theory

Using the expression for the average energy (49), the entropy is readily given in terms of the location of the event horizon r_+

$$S_{\rm BH} = \beta\langle E\rangle - I = \frac{\mathrm{Vol}(\mathbf{S}^{n-1})}{4G_{\rm N}} r_+^{n-1} \quad . \tag{53}$$

An important thing to note is that since the surface at r_+ is the area of the horizon $A = \mathrm{Vol}(\mathbf{S}^{n-1})r_+^{n-1}$, the Hawking-Bekenstein formula for the entropy is satisfied

$$S_{\rm BH} = \frac{A}{4G_{\rm N}} \quad .$$

The entropy of the black hole can be compared to the predictions for the field theory taken to be on the boundary in the sense explained in section 3.4. As mentioned in section 5.1, the boundary manifold of the asymptotic AdS_n spaces is $\mathbf{S}^{n-1}\times\mathbf{S}^1$. The system on $\mathbf{S}^{n-1}$ will be at high temperatures as $\beta \to 0$. As is evident from equation (42), the limit of $\beta \to 0$ can be taken two ways; either $r_+ \to 0$ or $r_+ \to \infty$. From the previous discussion, it is apparent that large r_+ is the most favourable configuration. This also means that the mass of the black hole is taken to be large. The limit given by equation (43) is therefore valid.

With this knowledge, one can express the entropy for the large black hole in quantities related to the field theory. It should be noted that this result is only valid for large effective coupling $g_sN \gg 1$. Restricting to the case of $n = 4$ and using the five dimensional Newton's constant given on the form (3.3), it becomes possible to write the result (53) for the entropy as

$$S_{\rm BH} = \frac{A}{4G_{\rm N}^{(5)}} = \frac{2\pi^2}{4}\frac{2N^2}{b^3}(\pi^3 b^6 T^3) = \pi^4 b^3 N^2 T^3, \quad g_sN \gg 1 \quad . \tag{54}$$

It is seen to have a N^2 dependence, which is true for all n. When comparing with the field theory, it is not possible to determine the constant of proportionality since this limit is valid for $g_sN \ll 1$. But since the theory only has one dimensionful parameter, namely the temperature, one can on dimensional grounds expect the energy scaling as the n'th power of the temperature. The entropy density on $\mathbf{S}^{n-1}$ therefore scales for small β as [30]

$$S_{\rm YM} \propto \frac{1}{\beta^{n-1}} \quad .$$

Then, due to the relation $\beta \sim 1/r_+$ given by equation (43) in the large r_+ limit, the entropy density of the boundary field theory is of the order r_+^{n-1}. This is how the area of the black hole horizon scales. The prediction is therefore asymptotic; that is, as $r_+ \to \infty$, the entropy density scales as a multiple of the horizon area exactly as was found by equation (53).

One can compute the entropy of the four dimensional SU(N) field theory in the limit $g_s N \ll 1$ [32]

$$S_{\rm YM} = \frac{4}{3}\pi^4 b^3 N^2 T^3, \quad g_s N \ll 1 \quad .$$

Thus, even though the entropy is valid for two different limits they still agree up to a factor of 4/3. It is conjectured that there exists a continuous function of $g_s N$ taking the value one at $g_s N \to \infty$ and 4/3 at zero. The first order correction to the function has been computed in both limits (see [1] or [32]).

The details of section 5.7 can now be met with the criterion for confinement / de-confinement from section 5.1. For temperatures lower than T_0, the field theory can not have a black hole dual theory. The dual is therefore the thermal AdS; that is, a space filled with a gas of particles at some temperature. The entropy or energy at these temperatures is of order $\mathcal{O}(1)$. The field theory at these temperatures must therefore be in the confined phase. When the temperature is greater than T_0, two black hole configurations become possible. Only the larger is locally stable, and in the range $T_0 < T < T_1$ the favourable configuration is still thermal AdS. The field theory is therefore still confined. The small black hole configuration is always an unfavourable configuration. Finally, when the temperatures exceed T_1, the larger black hole configuration will be most favourable. The entropy of this configuration is of order $\mathcal{O}(N^2)$ as seen from equation (54). It is therefore evident that the field theory goes from the confined phase to the deconfined phase. The deconfined field theory at this temperature is dual to the large black hole solution.

6 Conclusion

From the considerations of the extremal black p-brane solution of supergravity, the 3-brane was found to be particular interesting because it had a constant dilaton throughout spacetime and a finite size horizon. Its geometry showed a weakly curved behaviour for large values of $g_s N$. In section 3.2 the solution showed a decoupling into two systems in the low energy limit: (a) the horizon, whose geometry turned out to be the maximally symmetric $\mathrm{AdS}_5 \times \mathbf{S}^5$, describing the low energy physics as measured from infinity due to a redshift and (b) massless closed strings propagating in spacetime with negligible cross-section with the horizon. Secondly, a set of coincident D-branes placed in ten dimensional spacetime was considered. For weak-coupling where the branes behaved like rigid hyperplanes, their world volume dynamics could be described by a $\mathrm{U}(N)$ gauge theory for small values of the effective coupling $g_s N$. In particular, the four dimensional gauge theory living in the world volume of a set of coincident D3-branes was stated to be a conformal theory with a convenient topological expansion for a large number of degrees of freedom. In section 3.1 the low energy limit for the system was taken and a decoupling into two systems was found: (a) the low energy description on the branes, namely the conformal $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $\mathrm{SU}(N)$ and (b) massless closed strings propagating in the ten dimensional bulk. Based on the two low energy descriptions of the D-brane physics in their respective limits of $g_s N$ the AdS/CFT conjecture was established.

A discussion about the matching of coordinates of the field theory and the AdS_5 spacetime followed. It was found that the boundary of anti-de Sitter space corresponded to the UV limit of the field theory. From this discussion the notion of holography originated, namely that the physics in five dimensional anti-de Sitter backgrounds can be encoded in a four dimensional conformally invariant gauge theory. Interestingly, the choice of coordinates seemed to have an impact on the physics. However, this choice could be explained to be the same as choosing a specific regularisation scheme. In particular, two choices of coordinates were used. The Poincaré coordinates with the induced structure $\mathbb{R} \times \mathbb{R}^3$ at the boundary and the global coordinates with the $\mathbb{R} \times \mathbf{S}^3$ structure. It was shown that the two boundaries are conformally related.

Using the conjectured equivalence between the partition functions of the two theories, quantities could be explored in a tractable limit and compared to the dual theory in the opposite limit. The focus was set on performing computations on the gravitational side of the correspondence such that predictions of the strongly coupled gauge theory could be obtained. From the consideration of the non-extremal black 3-brane, the contributing Schwarzschild-AdS_5 black hole spacetime was found and the AdS/CFT correspondence was generalised to a duality between string theory on asymptotic AdS_5 spacetimes and a four dimensional guage theory. The path integral, which expresses the partition function, was evaluated using the saddle-point approximation in which only classical actions are included. For this, the generalised Einstein-Hilbert action with the Hawking-Gibbons surface term was considered. However, since both the bulk and surface action are divergent for the asymptotic AdS_5 spacetimes, some renormalisation scheme was required.

First, using the technique of holographic renormalisation, the action was renormalised by performing a minimal subtraction consisting of a finite number of counter terms that removed the divergent pieces near the boundary. The stress-energy tensor of the boundary could then be extracted from the renormalised action. Interestingly, the stress-energy tensor was shown to depend on the choice of regularisation, that is, the choice of coordinates. For the anti-de Sitter space, the choice of local coordinates lead to a vanishing stress-energy tensor, but for the choice of global coordinates it leads to a nonzero stress-energy tensor. This could be interpreted as a shift in energy. For the black hole metric, the energy was found to consist of both the contribution of the AdS metric and an additional contribution from the black hole correction. The new contribution is identified as the mass of the black hole.

Next, the partition function was approximated by the two competing spacetime configurations: (a) the anti-de Sitter space allowing thermal radiation and (b) the Schwarzschild-AdS black hole solution. Using the thermodynamical analogy of black holes, the temperature of the Schwarzschild-AdS black hole could be determined. This was a matter of determining the correct period of imaginary time such that the conical singularity was removed. At a given temperature T_0, it was shown that two branches of black hole radii became possible. Obtaining a finite action of the black hole solution by subtracting the AdS background, the various thermodynamical quantities of the black hole configuration could be determined. The free energy showed that the black hole branch of large radii at some temperature T_1 greater than T_0 is the more thermodynamical favourable configuration of spacetime. In addition, its specific heat turns out to be positive and it is therefore at least locally stable. One must expect that thermal radiation in AdS at some temperature T_2 undergoes a gravitational collapse such that the only possibility is the large black hole solution. Remarkably, the energy of the black hole computed from the action difference and the holographic renormalised action, respectively, was found to agree. However, the holographic method provides, in some sense, more information since it also includes the background energy. The findings were interpreted in the strongly coupled dual gauge theory as a transition from a confined to a de-confined phase.

The black hole entropy was determined in the limit of large mass for which the configuration was found to be stable and favourable. A comparison of scaling dependence of entropy between the two theories became possible since the only dimensionful scale of the strongly coupled field theory on the boundary $\mathbf{S}^{n-1}$ was the temperature. An agreement up to a fixed multiple of the black hole horizon area was found. At small $g_s N$, the entropy of the field theory was stated and seen to have the correct scaling, but a disagreement in the factor of proportionality of 4/3 compared to the large $g_s N$ result. This was completely expected since the results were obtained in two different limits.

Appendices

Induced metric (from [18]): Let M be an m-dimensional submanifold of an n-dimensional Riemannian manifold N with the metric g_N. If $f : M \to N$ is the embedding which induces the submanifold structure of M, the pullback map f^* induces the natural metric $g_M = f^* g_N$ on M. The components of g_M are given by

$$g_{M\mu\nu} = g_{N\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} \tag{55}$$

where f^α denotes the coordinates of $f(x)$.

A Maximally Symmetric Spacetimes

Consider a spacetime $(M, g_{\mu\nu})$ in the spirit of symmetries as introduced in section D.6. An n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors is referred to as a maximally symmetric space (see appendix C in [27] for a proof). A manifold with maximal symmetry has the special property of constant curvature $\mathcal{R}$ over its entire manifold. In general, such spaces can be classified by the dimensionality n, the curvature scalar $\mathcal{R}$, the metric signature, and some additional discrete information about the global topology. One can show that for maximally symmetric spaces there exists a unique relationship between the Riemann tensor and the metric [5] given by

$$\mathcal{R}_{\rho\sigma\mu\nu} = \frac{\mathcal{R}}{n(n-1)} \left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu} \right) \quad . \tag{56}$$

This relation also works as a way to check whether a metric is maximally symmetric or not. Say the Ricci curvature is constant, then if the Riemann satisfies the above relation at each point of the entire manifold then the metric will be maximally symmetric. The Ricci tensor can be obtained by taking the trace of the Riemann tensor

$$\mathcal{R}_{\mu\nu} = \frac{\mathcal{R}}{n} g_{\mu\nu} \quad . \tag{57}$$

This equation is exactly of the form of Einstein's vacuum equations with a cosmological constant. In the next section, spacetime solutions to this equation will be considered.

A.1 The Cosmological Constant

In absence of sources, the solutions of Einstein's vacuum equation $\mathcal{R}_{\mu\nu} = 0$ allow the flat Minkowski spacetime. However, in the presence of a cosmological constant,

$$\mathcal{R}_{\mu\nu} = \frac{2\Lambda}{n-2} g_{\mu\nu} \quad , \tag{58}$$

the field equations imply curved spacetimes. Spacetimes, which are solutions to this equation, are referred to as Einstein spaces. For $\Lambda = 0$, one recovers the vacuum equations, while the case of $\Lambda \neq 0$ leads to spacetimes of maximal symmetry. Comparing equation (57) with (58), the field equations in four dimensions give $\mathcal{R} = 4\Lambda$.

Much like one uses the assumption of spherical symmetry to derive the Schwarzschild metric and Birkhoff's theorem to determine its uniqueness (see section (D.1)), one can derive the Schwarzschild-(anti)-de Sitter space metric as a consequence of spherical symmetry. There exist generalised theorems, which state that this is the unique solution to Einstein's vacuum equations with a non-zero cosmological constant [21]. In four dimensions, a manifold has spherical symmetry if the dimension of the Killing algebra equals three; that is, there exist three rotational Killing vectors, which close under commutation. The Schwarzschild-(anti)-de Sitter solution turns out to have a similar form as the Schwarzschild solution. In four dimensions, the Schwarzschild(anti)-de Sitter space for positive and negative cosmological constant, respectively, can be written in the static form [22]

$$\begin{aligned} ds^2 &= -V dt^2 + V^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ V(r) &= 1 + \frac{r^2}{b^2} - \frac{2MG_{\mathrm{N}}}{r}, \quad b^2 = -\frac{3}{\Lambda} \quad , \end{aligned} \tag{59}$$

with $(t, r, \theta, \phi) \in \mathbb{R} \times [0, \infty[\times \mathbf{S}^2$. These coordinates are global meaning that they cover the manifold entirely. The roots of the function $V(r)$ determines the location of an event horizon, say $r = r_+$ satisfying $V(r_+) = 0$. This means that the space has a black hole. See the explicit calculation in the case of four dimensions in section 5.2. The constant b is denoted the radius of curvature while M is the mass of the black hole. When $M = 0$, the metric reduces to anti-de Sitter space without a black hole. This space has a timelike Killing vector defined everywhere contrary to the case of the Schwarzschild-AdS space. For the general case of non-zero mass, the metric is seen as $r \to \infty$ to be asymptotic to anti-de Sitter.

For a general dimension, the solution (59) is normalised such that relation between the radius of curvature and the Ricci curvature is

$$\mathcal{R} = -\frac{n(n-1)}{b^2} \propto -\frac{1}{b^2} \quad , \tag{60}$$

which fixes the cosmological constant

$$\Lambda = -\frac{(n-1)(n-2)}{2b^2} \propto -\frac{1}{b^2} \quad .$$

Locally, a maximally symmetric space of a given dimension and signature is fully specified by $\mathcal{R}$. For Euclidean signature, the classification of maximally symmetric spaces is simply whether $\mathcal{R}$ is positive, zero, or negative, corresponding to a plane, a sphere, and a hyperboloid, respectively. The maximally symmetric manifolds for the Lorentzian signature are classified locally by the sign of Λ, which is constant over the entire manifold. For vanishing curvature $\Lambda = 0$, one has the Minkowski spacetime, while the case of positive Λ is called the de Sitter space, and finally for negative Λ, one has the anti-de Sitter space.

A.2 Anti-de Sitter Spacetime

One relates the anti-de Sitter space with the negative curved spacetime, $\mathcal{R} < 0$. As seen from equation (59), the space has no coordinate singularity, no horizon and using the definitions from section (D.6) it is possible to show that it is static.

First note that the form of the anti-de Sitter metric given by equation (59) with $m = 0$ is independent of time, hence $\xi^\mu = [1, 0, 0, 0]$ is a timelike Killing vector field. Those integral curves are assumed to be complete. (This is shown later via the existence of conjugate points.) Lowering the index with the metric does not affect the space indices. The fact that only the time component is non-zero together with the totally antisymmetry of the equation $\xi_{[\alpha}\nabla_\beta \xi_{\gamma]}$ results in that each plus term equals each minus term and therefore all equations are zero.

Other interesting coordinate choices exist for the anti-de Sitter metric which have significance for higher dimensions. An n-dimensional anti-de Sitter space can be represented by embedding an n-dimensional hyperboloid

$$-x_0^2 - x_n^2 + \sum_{i=1}^{n-1} x_i^2 = -b^2 \tag{61}$$

in an $(n+1)$-dimensional flat pseudo-Riemannian manifold $\mathbb{R}^{2,n-1}$ with metric

$$ds^2 = -dx_0^2 - dx_n^2 + \sum_{i=1}^{n-1} dx_i^2 \quad . \tag{62}$$

Anti-de Sitter space has the topology $\mathbf{S}^1 \times \mathbb{R}^{n-1}$ and is said to hold one time dimension, while the rest is regarded as spatial dimensions. It is homogeneous, isotropic, and has the isometry group $\mathrm{SO}(2, n-1)$ by construction, that is

$$\mathrm{SO}(2, n-1) = \{M \in \mathrm{SL}(n+1, \mathbb{R}) \mid M^t \eta_{2,n-1} M = \eta_{2,n-1}\} \quad . \tag{63}$$

One can obtain the induced metric by choosing a suitable set of coordinates that solves the embedding condition. These coordinates must be smooth maps satisfying the properties given by equation (55). In the following, two such choices are explored. From now on, an n-dimensional anti-de Sitter space is denoted AdS_n and when discussed on general terms simply as AdS.

A.3 Global Coordinates

Equation (59) stated the metric of AdS in global coordinates without further explanation. However, it can actually be derived from a specific choice of an embedding that solves the condition (61). In this section, three forms of global coordinates shall be reviewed, starting by the embedding

$$\begin{aligned} x_0 &= b \cosh\rho \cos\tau \\ x_n &= b \cosh\rho \sin\tau \\ x_i &= b \sinh\rho\, \Omega_i, \quad i = 1, ..., n-1 \quad , \end{aligned} \tag{64}$$

where the Ω_i functions must satisfy the $(n-2)$-sphere condition

$$\sum_{i=1}^{n-1} \Omega_i^2 = 1 \quad .$$

Notice that the set of functions $\{\Omega_i\}$ only depends on the sphere-coordinates. Substituting the choice into equation (61)

$$\begin{aligned} b^2 &= b^2 \left(\cosh^2\rho \left(\cos^2 t + \sin^2 t\right) - \sinh^2\rho \left(\sum_i^{n-1} \Omega_i^2 \right) \right) \\ &= b^2 \left(\cosh^2\rho - \sinh^2\rho\right) \quad . \end{aligned}$$

The embedding can be used to compute the induced metric on the hyperboloid

$$ds^2 = b^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{n-2}^2\right) \quad . \tag{65}$$

This can be seen by using equation (55) and the $(n+1)$-dimensional flat pseudo-Riemannian metric

$$\begin{aligned} g_{\tau\tau} &= b^2 \left(-\cosh^2\rho \sin^2\tau - \cosh^2\rho\cos^2\tau\right) d\tau^2 = -b^2\cosh^2\rho\, d\tau^2 \\ g_{\rho\rho} &= b^2 \left(-\sinh^2\rho\cos^2\tau - \sinh^2\rho\sin^2\tau + \cosh^2\rho \sum_i^{n-1} \Omega_i \right) d\rho^2 = d\rho^2 \\ g_{\phi_i\phi_i} &= \left(\sum_j^{n-1} \frac{\partial x_j}{\partial\phi_i} \right) d\phi_i^2 = b^2 \left(\sinh^2\rho \sum_j^{n-1} \frac{\partial\Omega_j}{\partial\phi_i} \right) d\phi_i^2, \quad i = 1...n-2 \quad , \end{aligned}$$

where the ϕ_i's denote the angular coordinates on the unit $(n-2)$-sphere. The sum over the last terms reveals the last part of equation (65)

$$\sum_{i}^{n-2} g_{\phi_i\phi_i} = b^2 \sinh^2 \rho \, d\Omega_{n-2}^2 \quad .$$

By choosing $\rho \geq 0$ and $0 \leq \tau < 2\pi$, the embedding given by equation (64) covers the entire hyperbolid once and the coordinates (ρ, τ, Ω_i) are therefore called global. Noting that the coordinate τ and $\tau + 2\pi$ for fixed $\{\rho, \Omega_i\}$ represent the same place on the hyperboloid, it is evident that the embedding have closed timelike curves with period $2\pi b$. This is what the $\mathbf{S}^1$ in the topology structure represents.

In the case of having two timelike coordinates, it is, however, possible to eliminate the timelike closed curves by passing to the universal covering space. This is often described as unrolling the embedding and is the spacetime with the metric given by (65) where the range of τ is allowed to be from $-\infty$ to ∞. One must remember that the timelike curves are not an intrinsic property of the spacetime and must be regarded as being an artifact of the specific choice of embedding (see [5]). It is this universal covering space one refers to being AdS. With this identification, the topology is $\mathbb{R}^{1,n-1}$ and the isometry group is a cover of $\mathrm{SO}(2, n-1)$. Furthermore, the maximal compact subgroup of $\mathrm{SO}(2, n-1)$ is $\mathrm{SO}(2) \times \mathrm{SO}(n-1)$.

Now, one can perform the following coordinate transformation

$$r' = \sinh \rho \quad \Rightarrow \quad \begin{aligned} & d\rho = \frac{1}{\cosh \rho}\, dr' = \frac{1}{\sqrt{1 + r'^2}}\, dr' \\ & \cosh^2 \rho = 1 + r'^2 \quad , \end{aligned}$$

where the range of ρ is carried over to $r' \geq 0$. Therefore, equation (65) takes the form

$$ds^2 = b^2 \left[-\left(1 + r'^2\right) d\tau^2 + \left(1 + r'^2\right)^{-1} dr'^2 + r'^2 \, d\Omega_{n-2}^2 \right] \quad . \tag{66}$$

Finally, to get it on the form of equation (59) with $M = 0$, one can perform the coordinate transformation

$$r = br' \quad \wedge \quad t = b\tau \quad ,$$

such that

$$\begin{aligned}\left(\frac{b^2}{1+r'^2}\right) dr'^2 &= \left(\frac{b^4}{r^2+b^2}\right) dr'^2 = \left(\frac{b^2}{r^2+b^2}\right) dr^2 \\ -b^2\left(1+r'^2\right) d\tau^2 &= -b^2\left(1+\frac{r^2}{b^2}\right) d\tau^2 = -\left(1+\frac{r^2}{b^2}\right) dt^2 \quad ,\end{aligned}$$

and $b^2 r'^2 = r^2$ gives

$$ds^2 = \left(1+\frac{r^2}{b^2}\right) dt^2 + \left(1+\frac{r^2}{b^2}\right)^{-1} dr^2 + r^2\, d\Omega_{n-2}^2 \quad . \tag{67}$$

This is the desired form given by (59) with zero mass.

A.4 Poincaré Coordinates

A different embedding, which turns out only to cover half of the hyperboloid, but has some other desired properties, is the Poincaré coordinates $(u, t, \vec{y})$ given by

$$\begin{aligned}x_0 &= \frac{1}{2u}\left[1+u^2(b^2+\vec{y}^2-t^2)\right] \\ x_d &= but \\ x_i &= buy_i, \quad i = 1, ..., d-2 \\ x_{d-1} &= \frac{1}{2u}\left[1-u^2(b^2-\vec{y}^2+t^2)\right] \quad ,\end{aligned}$$

where the range of the coordinates becomes $u > 0$, $t \in \mathbb{R}$, and $\vec{x} \in \mathbb{R}^{d-2}$. Since the coordinates solve the embedding condition (61), the induced metric can be obtained similarly to the calculation done for the global coordinates

$$\begin{aligned}ds^2 &= b^2\left[\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{y}^2)\right] \\ &= b^2\left[\frac{du^2}{u^2} + u^2 \eta_{\mu\nu} dx^\mu dx^\nu\right] \quad .\end{aligned}$$

One can simplify this a little further by introducing the coordinate $u = \frac{1}{z}$, thus obtaining the conformal compactification

$$ds^2 = \frac{b^2}{z^2}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right] \quad . \tag{68}$$

This form shows one of the desired properties of this embedding, namely that the conformal boundary at $z = 0$ has the same metric as flat Minkowski space

$$\eta_{\mu\nu}dx^\mu dx^\nu \quad .$$

As will be discussed in section 2.1, this boundary spacetime will play a special role when considering conformal field theories.

A.5 Additional Mapping

In section A.3, a map from the flat space metric given by (62) with the embedding condition (61) to the AdS metric, given in the global coordinates by equation (59) with which $M = 0$, was established. For completeness, the inverse map is given in this section. Writing the flat metric in polar coordinates by introducing the radial coordinate $r^2 = \sum_{i=1}^{n-1} x_i^2$, one has

$$-x_0^2 - x_n^2 + r^2 = -b^2$$

$$ds^2 = -dx_0^2 - dx_n^2 + dr^2 + r^2 d\Omega_{n-2} \quad .$$

The required mapping between anti-de Sitter space and the hypersurface is the following [22]

$$\begin{aligned} x_0 &= \sqrt{b^2 + r^2}\cos\frac{t}{b} \\ x_n &= \sqrt{b^2 + r^2}\sin\frac{t}{b} \quad , \end{aligned}$$

which is seen to satisfy the embedding condition. The radial coordinate and sphere are carried over unchanged while the $-dx_0^2 - dx_n^2$ part gives the $t - t$ component and a contribution to the $r - r$ component in equation (59). The coordinates are one-to-one except for t, which with a period of $2\pi b$ maps to the same point.

A.6 The Causal Structure

When studying the casual structure of a spacetime, it is often practice to perform a conformal compactification. A particularly simple form of the conformal diagram multiplied by a conformal factor can be obtained by performing the coordinate transformation of the global coordinates given by equation (65)

$$\sinh\rho = \tan\chi \quad \text{or} \quad \cosh\rho = \frac{1}{\cos\chi}, \quad 0 \leq \chi \leq \frac{\pi}{2} \quad ,$$

where the range of χ holds for $n > 2$ dimensions and for $n = 2$ the range is $-\frac{\pi}{2} \leq \chi \leq \frac{\pi}{2}$. The coordinate transformation takes the metric to the form

$$ds^2 = \frac{b^2}{\cos^2\chi}\left(-dt^2 + d\chi^2 + \sin^2\chi \, d\Omega_{n-2}^2\right) \quad . \tag{69}$$

The coefficient in front is a conformal factor, so multiplying by the inverse does not change the conformal structure of the spacetime (see [5] appendix G). The remaining expression for the metric is the metric of half the Einstein static universe due to the range of χ. Thus, the conformal diagram can be viewed as the infinite strip on a cylinder in the range of χ.

The boundary of AdS_n is located at $\chi = \frac{\pi}{2}$ and has a second order pole there. One therefore can not induce a metric. To obtain a metric, one must pick a positive function e.g. $f = \cos\chi$ and evaluate

$$g_{(0)} = f^2 g|_{\mathbb{R}\times\mathbf{S}^{n-2}} \quad , \tag{70}$$

where the boundary structure has been indicated. The metric $g_{(0)}$ depends on the choice of the defining function. Different functions are related by $f' = fe^u$, which implies that $g_{(0)}$ is defined up to a conformal transformation. The AdS metric therefore induces a conformal structure at the boundary.

The boundary structure of the form given by equation (67) is of interest. Taking a radial slice at infinity ($r \gg b$), one again obtains the structure $\mathbb{R} \times \mathbf{S}^{n-2}$ multiplied by a conformal factor

$$ds^2 = \frac{r^2}{b^2}\left[-dt^2 + b^2 \, d\Omega_{n-2}^2\right] \quad . \tag{71}$$

A.7 Geodesics in Global Coordinates

Particles and fields move along geodesics curves. In the case of anti-de Sitter space, massive particles can never reach infinity, but refocuse with a specific period as shall be shown. For massless particles it is possible to impose a boundary condition at infinity such that the incoming and outgoing flux are equal. They can then reach infinity and come back in finite time.

One feature the AdS space exhibits is that every complete timelike geodesic has at least one pair of conjugate points (p, q). That is, the geodesic intersects some point p and eventually intersects some other point q. (The strict definition can be found in section 9.3 of [27].)

According to [27] Proposition 9.3.2, it should be sufficient to show that AdS_n satisfies the timelike generic condition and $R_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ for all timelike ξ^μ. The timelike generic condition is said to be satisfied if each timelike geodesic has at least one point for which $\mathcal{R}_{\rho\sigma\mu\nu}\xi^\rho\xi^\mu \neq 0$. The Riemann curvature of AdS_n is given in equation (56). The condition can therefore be written

$$\begin{aligned} 0 &\neq \mathcal{R}_{\rho\sigma\mu\nu}\xi^\rho\xi^\mu = \kappa\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right)\xi^\rho\xi^\mu \\ &= \kappa\left(g_{\sigma\nu}\xi^\rho\xi_\rho - \xi_\nu\xi_\sigma\right) = \kappa\left(g_{\sigma\nu}\xi^\rho\xi_\rho - g_{\nu\lambda}\xi^\lambda\xi_\sigma\right) \quad , \end{aligned}$$

where $\kappa \propto \mathcal{R}$. By definition, AdS_n space has $\kappa < 0$, hence

$$\begin{aligned} g_{\sigma\nu}\xi^\rho\xi_\rho &\neq g_{\nu\lambda}\xi^\lambda\xi_\sigma \\ n\xi^\rho\xi_\rho &\neq g^{\sigma\nu}g_{\nu\lambda}\xi^\lambda\xi_\sigma \\ n\xi^\rho\xi_\rho &\neq \xi^\sigma\xi_\sigma \quad . \end{aligned}$$

Since the timelike condition states $\xi^\mu\xi_\mu < 0$, the condition is satisfied for all timelike geodesics if $n \neq 1$. For the second condition, the Ricci tensor is given by equation (57)

$$R_{\mu\nu}\xi^\mu\xi^\nu = 3\kappa g_{\mu\nu}\xi^\mu\xi^\nu > 0 \quad , \tag{72}$$

which is strictly positive, since $\kappa < 0$ and $\xi^\mu \xi_\mu < 0$. Both conditions are thus satisfied showing that every complete timelike geodesic has a pair of conjugate points. In addition, one can similarly see from equation (72) that the null generic condition is not satisfied and therefore, based on Proposition 9.3.7 in [27], there is not sufficient information to say whether every complete null geodesic possesses a pair of conjugate points or not.

The geodesics in anti-de Sitter space have been investigated in detail in [11]. Here, a simple analysis will be performed to obtain a differential equation for the radial coordinate. The geodesic equation reads

$$\frac{dx^2}{ds^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \quad ,$$

where $\Gamma^\rho_{\mu\nu}$ is the metric connection with respect to the metric given in the coordinates by equation (27) and s is an affine parameter. One has the condition

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \epsilon \quad , \tag{73}$$

where ϵ is one for massive particles and zero for massless particles. Since the AdS has spherical symmetry, one can restrict attention to the equatorial plane $\theta = \frac{\pi}{2}$ without loss of generality. The timelike and rotational Killing vectors are given by

$$\xi^\mu = (\partial_t)^\mu = [1, 0, 0, 0], \quad \psi^\mu = (\partial_\phi)^\mu = [0, 0, 0, 1] \quad .$$

Lowering the index gives

$$\begin{aligned} \xi_\nu &= g_{\mu\nu} \xi^\mu = -V \\ \psi_\nu &= g_{\mu\nu} \psi^\mu = r^2 \sin\theta = r^2 \quad . \end{aligned}$$

As mentioned in section D.6, Killing vector fields satisfy $\xi_\mu \dot{x}^\mu = \text{const.}$, thus

$$\begin{aligned} E &= -\xi_{mu} \dot{x}^\mu = -V \frac{dt}{ds} \\ L &= -\psi_\mu \dot{x}^\mu = r^2 \frac{d\phi}{ds} \quad , \end{aligned}$$

where the minus in the definition of E is conventional since it is associated with energy. Inserting these constants into the normalisation condition given by equation (73), one finds

$$\left(\frac{dr}{ds}\right)^2 = E^2 + \left(\epsilon - \frac{L^2}{r^2}\right) V(r) \quad ,$$

which is exactly the form obtained in the case of a flat space Schwarzschild metric where $V(r) = 1 - 2M/r$. However, in the case of AdS, the presence of the negative cosmological constant in the function V alters the behaviour of the geodesics significantly.

A.8 Physics and Initial Configurations

Anti-de Sitter space differs from the flat and positive curved de Sitter space in that it does not possess a Cauchy surface, that is, a spacelike surface for which its domain of dependence equals the entire manifold. This fact means that the theorems of well-posed initial value formulations are not enough to put physics onto the spacetime. It is necessary to impose both an initial configuration and specify boundary conditions which describes the radiation from infinity.

B String Theory

As was emphasized in the introduction, string theory is the foundation on which the AdS/CFT correspondence emerges. This section will give provide a brief introduction and will summarise the essential results of type IIB superstring theory.

Based on previous success of investigating conflicting theories new coherent theories have been discovered. This process of unification has revealed an enormous amount of insight and have resulted in the present recipe of obtaining a unified description of all fundamental interactions. Namely, it is formulated as the task of combining the classical theory of relativity with the theory of quantum fields. This is two theories with two very different applications. General relativity is the tool for understanding cosmic scale physics. It postulate the existence of a spacetime for which the curvature is related to the matter distribution. Hence, gravity is the dynamics of spacetime. Quantum field theory is the (special) relativistic version of quantum mechanics and is the tool for understanding microscopic physics. It associates a particle to every field in the theory; both matter and force. The gravitational force is represented by the particles known as gravitons propagating on a fixed background. The seemingly difference between these two view points gives severe complications when trying to combine them. String theory succeeds in overcoming these difficulties and provides a consistent framework for the theory of quantum gravity.

The main concept is to introduce a fundamental string. A string spans one spatial dimension and propagates in time. For each vibrational mode of the microscopic string one identifies a particle. It is a general feature that one of these vibrational states is the graviton. This is significant because it indicate that general relativity arises naturally within the framework of a quantum theory. Another property of string theory that is needed for consistency is local supersymmetry. Supersymmetry is a purely theoretical idea about the symmetry between bosons and fermions, but plays dominant role which gives rise to superstring theory. It is roughly put into play by introducing fermionic coordinates on the world-sheet of the string.

Being a candidate to the theory of everything uniqueness is of great importance. Two key results provides indication of uniqueness. First, unlike other theories the dimension of spacetime is calculated rather than postulated. A calculation shows that in order for a superstring theory to be consistent the dimension of spacetime must be equal to ten. At least the flat ten dimensional space exists in perturbation theory. Second, the theory contains no adjustable dimensionless parameters. It is therefore not possible to obtain a continuum of different theories by varying parameters. However, other theories could be reached by means like dualities.

Relativistic strings can be explored perturbatively by letting them propagate on a flat background. When the string is quantized one finds that the spectrum consist of a massless sector accessible at low energy and a infinite tower of massive excitations relevant for high energies. These excitations are separated by a gap determined by the string tension. However, being a theory of gravity one would like string theory non-perturbatively to determine the background upon which it is propagating. In fact, this is not possible considering only strings and it turns out that string theory is not a theory based solely on strings, the theory actually contains other spatially extended objects as well. These are known as D-branes and arise from non-perturbative excitations. Considering the tension of the D-branes one finds that they can become arbitrary lighter than the fundamental string with increasing string coupling. Therefore they can dominate the low energy physics. Furthermore, T-duality and S-duality reveals that there exist precise mappings between weakly coupled theories and strongly coupled theories. Some superstring theories even grow an additional dimension for which eleven dimensional supergravity is revealed. There exist five known superstring theories and one eleven dimensional supergravity. Together they span a web of dualities and form the corners of the one theory known as M-theory. In string theory one works with three perspectives: the world-sheet, the spacetime, and the brane perspective.

The Type IIB superstring theory plays for various reasons the dominant role in the AdS/CFT correspondence considered in this text. Primarily, as mentioned in section B.4, because it contains a D3-brane which supergravity solution have special properties like finite area of the horizon, constant dilaton, and a highly symmetric near-horizon geometry. In this section some aspects of the supersymmetric string theories will briefly be introduced for the purpose of reference with a primary focus on the Type IIB superstring theory. Type II superstring theories are theories of oriented closed strings.

B.1 The Massless Content

To construct a supersymmetric theory of open strings one introduces a set of dynamical anti-commuting variables on the world-sheet of the string. These fermionic coordinates describe a world-sheet fermionic field which state space quantization leads to two different set of boundary conditions of the fermionic coordinates. These two possible choices are what breaks the state space into the Ramond(R) sector and the Neveu-Schwarz(NS) sector (see e.g. [32]). Both sectors contain an infinite tower of states. The R sector also shows supersymmetry on the world-sheet. Spacetime supersymmetry arises when combining states from the R and NS sectors which is exactly what one does when constructing a theory of open strings. The closed string spectrum is obtained by combining two copies of the open string spectrum one for the right-moving modes and one for the left-moving modes. There exist two inequivalent ways of doing this leading to the two type II superstring theories.

In particular, one is interested in the massless spectrum which is dominating the low energy limit where supergravity is an appropriate description of sting theory. The multiplet is therefore known as the supergravity multiplet. The massless content of the type IIB superstring theories are given by the $\mathcal{N} = 2$ supersymmetry algebra. For type IIB the massless spectrum is readily described in terms of the stability group SO(8) arising from the Lorentz symmetry SO(9,1). Taking the same projection for the left and right side modes leads to the massless spectrum [17]

$$(\mathbf{8_v} \otimes \mathbf{8_s}) \otimes (\mathbf{8_v} \otimes \mathbf{8_s}) \quad .$$

Thus the same type of spinor is chosen for the right and left moving sectors making type IIB a chiral theory. Expanding the product of the vector representations and the spinor representations one obtains the NS-NS sector and the R-R sector, respectively,

$$\begin{aligned} \mathbf{8_v} \otimes \mathbf{8_v} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = \Phi \oplus B_{\mu\nu} \oplus G_{\mu\nu} \\ \mathbf{8_s} \otimes \mathbf{8_s} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+ = [0] \oplus [2] \oplus [4]_+ \quad , \end{aligned}$$

where $\mu, \nu = 0, ..., 9$. The NS-NS sector consists of the dilaton Φ, the antisymmetric two-form $B_{\mu\nu}$, and the symmetric traceless graviton $G_{\mu\nu}$. For the R-R sector $[n]$ denotes the n-times antisymmetrised representation of SO(8). More precisely $[0] = C_0$, $[2] = C_2$, and $[4]_+ = C_4^+$. The [4] is chosen to be the self-dual. The NS-NS and R-R sector constitute the bosonic components of type IIB supergravity. The fermionic components are given by the NS-R and R-NS sector

$$\mathbf{8_v} \otimes \mathbf{8_s} = \mathbf{8_c} \oplus \mathbf{56_s} = \psi^I_{\mu\alpha} \oplus \lambda^I_\alpha, \quad I = 1, 2 \quad ,$$

where $\alpha = 1...16$, since spinors has 16 components in ten dimensions. The representation $\mathbf{56_s}$ describes the gravitino and the representation $\mathbf{8_c}$ describes the dilatinos. As noted, the type IIB is chiral there are therefore two copies of both.

B.2 Double Expansion

Type IIB string theory has two parameters: the string length ℓ_s and the string coupling g_s. However, the string coupling is actually dynamically determined by the expectation value of the background dilaton field

$$g_s = e^{\langle\Phi\rangle} \quad .$$

Considering actions for strings reveals that the open string coupling and closed string coupling has the relation $g_s = g_{\text{open}}^2$. The open string coupling is also identified with the Yang-Mills coupling g_{YM} since the gauge theory bosons on Dp-branes is a theory of massless open strings. Quantities in string theory is often given in terms of the Regge slope parameter α'. This descends from the Regge trajectories of a rotating string and has the relation to the string length

$$\ell_s = \sqrt{\alpha'} \quad .$$

From which it is evident that α' has dimensions of length squared. The two parameters makes it possible to view string theory as a simultaneous expansion in two parameters.

- An expansion in α' controls the stringy excitations about the point-particle limit discussed under the supergravity section B.3. It is the expansion which corresponds to the quantum-mechanically treatment of the string world sheet.
- The second expansion is controlled by the string coupling g_s. This is the expansion in the number of string loops or equivalent genus of the string world sheet. $g_s \to 0$ corresponds to the weak-coupling limit while $g_s \to \infty$ corresponds to the strong-coupling limit. For type IIB superstring theory these two regimes are related by S-duality. Also, weak-coupling corresponds to classical string theory.

B.3 Supergravity Actions

The fundamental string tension is inverse proportional to α'. In the limit of large string tension $\alpha' \to 0$, it is possible to approximate string theory with a supergravity theory. Such a theory only takes the interactions between the modes of the massless spectum into account. The reason why this approximation works is that the massive modes of the spectrum becomes too heavy to be observed. One way to make sense of the limit is to consider a Minkowski background where the only dimensionless parameter is given by $\sqrt{\alpha'}E$. This means that the limit corresponds to a theory at low energy. The effective supergravity theories are non-renormalisable, but at low energy the higher order quantum corrections can be ignored in most cases. Another important thing to note is that the local supersymmetry makes sure that the actions are uniquely determined at least up to a normalisation constant.

Unlike the supergravity action for type IIA which can be obtained by performing a dimensional reduction from the eleven dimensional supergravity, an action for the type IIB theory must be constructed from the supersymmetric equation of motions. The action is then wrote down in such a way that it reproduces these equations. The bosonic part of the type IIB supergravity action is constituted of three parts [4]

$$S = S_{\text{NS}} + S_{\text{RR}} + S_{\text{CS}} \quad .$$

The first term accounts for the massless NS sector fields: the graviton, the dilaton, and the antisymmetric two-form with field strength $H_3 = dB_2$

$$S_{\text{NS}} = \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left[\mathcal{R} + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|H_3|^2 \right] \quad .$$

The second term accounts for the massless R-R sector fields

$$S_{\text{RR}} = -\frac{1}{4\kappa_0^2} \int d^{10}x \sqrt{-g} \left[|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right] \quad ,$$

where the field strengths are given by $F_{n+1} = dC_n$ and the true gauge invariant combinations are given by

$$\begin{aligned}\tilde{F}_3 &= F_3 - C_0 \wedge H_3 \\ \tilde{F}_5 &= F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \quad .\end{aligned}$$

The last part is a Chern-Simons term similarly to topological field theories

$$S_{\rm CS} = -\frac{1}{4\kappa_0^2}\int C_4 \wedge H_3 \wedge F_3 \quad .$$

Note that the wedge products adds up to a ten-form. There is, however, one important complication arising from the self-dual nature of the field strength $\tilde{F}_5$. As the term, included in the action, stands it does not impose the self-duality constraint and therefore the action describes twice as many degrees of freedoms as desired. There are various ways to overcome this problem, but one way is to impose the self-duality condition as an extra constraint

$$\tilde{F}_5 = \star\tilde{F}_5 \quad .$$

A final thing to mention is that one typically works with two frames of actions. The form of the action discussed in the above is called the string frame action. However, it is possible to rewrite it to a form known as the Einstein frame action which essential feature is that Ricci scalar is not multiplied by a factor involving the dilaton. This also leads to the identification [17]

$$2\kappa^2 = 2\kappa_0^2 g_s^2 = 16\pi G_N = (2\pi)^7 \alpha'^4 g_s^2 \quad . \tag{74}$$

The ten dimensional Newton constant $G_{\rm N}$ is thus determined dynamically by the dilaton. When one works in the weak-coupling limit where $g_s \ll 1$ the low energy field equations of the supergravity action can be trusted. The solutions are known as extended objects called p-branes. In the text special cases of these solutions shall be considered.

B.4 The Brane Spectrum

String theory is a theory of one-dimensional strings and other higher spatial extended objects, called branes. The stable branes in a given theory can be found by considering the massless particle spectrum. In particular the gauge fields in the R-R sector can couple to higher dimensional objects, called D-branes. The definition of D-branes, at least in the perturbative regime, is that open strings ends on them. Consequently, a string which do not end on a D-brane must be a closed string. The motivation for introducing D-branes is usually given in terms of T-duality. Though, this is not of interest here, the key is to consider the imposed boundary condition on the ends of the open strings. The only boundary condition which is compatible with Poincaré invariance is of the Neumann type. However, through T-duality the Neumann boundary condition inevitable introduces Dirichlet boundary conditions, which means the open string can end on a hypersurface that breaks Poincaré invariance. This hypersurface turns out to be a D-brane. A D-brane is a physical object which is extended in p spatial directions. When time is included, the dimension of the world volume of a Dp-brane is $p+1$ with the symmetry group

$$\mathbb{R}^{p+1} \times \mathrm{SO}(p,1) \times \mathrm{SO}(d-p-1) \quad , \tag{75}$$

where d is the spactime dimension. The factor $\mathbb{R}^{p+1} \times \mathrm{SO}(p,1)$ is the Poincaré group (translations and Lorentz group). The symmetry of the D3-brane is thus $\mathbb{R}^4 \times \mathrm{SO}(3,1) \times \mathrm{SO}(6)$. Considering point-like sources at the origin in the generalised Maxwell equations arising from a general q-dimensional gauge potential one can deduce what D-branes the massless gauge fields gives rise to, one finds

$$\begin{aligned} &\text{Electrically:} && q-1 \\ &\text{Magnetically:} && d-q-3 \quad . \end{aligned}$$

The type IIB brane spectrum is thus D(-1), D1, and D3 with magnetic dual D7, D5, and D3, respectively. Note that the D3-brane is self-dual. The NS two-form gives rise to the fundamental string F1 which have the magnetic dual known as NS5.

D-branes are objects with well-defined mass and charge per unit volume. It is possible to determine the tension of the D-branes by considering the propagation of a closed string between two branes. This involves considering the closed string as open strings such that one can employ string perturbation theory. The result is

$$T_{\mathrm{D}p} = \frac{1}{(2\pi)^p l_s^{p+1} g_s}, \quad T_F = \frac{1}{2\pi l_s^2}, \quad \text{and} \quad T_{\mathrm{NS5}} = \frac{1}{(2\pi)^5 l_s^6 g_s^2} \quad .$$

The tension of the fundamental string and its magnetic dual, the NS5-brane, is listed for comparison.

D-branes allow a way to introduce non-abelian gauge symmetries in string theory and this way gauge fields appear on their world volumes. Placing a brane in spacetime gives rise to a bosonic content consisting of one gauge field living in $p{+}1$ dimensions with $p{-}1$ degrees of freedom and $d{-}p{-}1$ scalars. The scalars are Goldstone modes resulting from spontaneous breaking the Poincaré invariance. The expectation values of the scalars roughly determines the position of the brane in the $d-p-1$ transverse dimensions. In fact, for the case of a U(1) symmetry the scalar does exactly determine the position of the brane, but for higher dimensional gauge symmetries it is only possible to diagonalise one of the generators at a time. One can think of this as the uncertainty principle in quantum mechanics.

The D-brane is a dynamical object which is affected by gravity. As the open strings that end on them, they respond to the different background fields in the theory. There is many contributing effective actions for the dynamics on D-branes, but the dominant world volume action is given by the DBI action [17]. For N D-branes the dynamics become more complicated and one will have to look at non-abelian extensions of this action. Although, non-abelian gauge theories will be of interest in this text, the understanding of these extensions are not of priority here. Each brane can also be realized as a solution in supergravity, see section B.3, and the geometry of these solutions is addressed in section 2.1.

B.5 S-duality

The duality known as S-duality relates a string theory with coupling constant g_s to a different string theory with coupling constant $1/g_s$.

Here the result that both the gauge theory and the string theory have SL(2, $\mathbb{Z}$) self-duality symmetry are discussed. There are two important things to note from the present discussion of S-duality. First, the connection between the symmetry of the two theories and second that S-duality of type IIB superstring relates it to itself.

For the field theory of interest, the $\mathcal{N} = 4$ super Yang-Mills theory, the electric-magnetic duality known from Maxwell's equations generalises to non-abelian gauge fields. The theory has a SL(2,$\mathbb{Z}$) duality under which the complex coupling constant

$$\tau = \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi} \tag{76}$$

transform as a modular parameter. The duality $\tau \rightarrow -1/\tau$ simplifies in the special case of $\theta = 0$ to the electric-magnetic duality given by

$$g_{\text{YM}} \rightarrow \frac{4\pi}{g_{\text{YM}}} \quad . \tag{77}$$

The S-duality transformation of type IIB superstring theory is realised through the complex field

$$\tau = \frac{i}{g_s} + \frac{\chi}{2\pi} \quad , \tag{78}$$

where χ is the expectation value of the R-R scalar C_0. Here the string coupling is written explicitly, but as mentioned it has a relation to the dilaton $g_s = e^{\langle \Phi \rangle}$. It is therefore evident that the duality transformation changes the sign of the dilatons under $\tau \to -1/\tau$ with $\chi = 0$. This is $g_s \to 1/g_s$ and is known as S-duality which in this case relates type IIB superstring theory to itself. This particular self-duality is worth noting in the discussion of validity of the AdS/CFT correspondence, section 3.

C Statistical Mechanics

Thermodynamics is the suitably framework for describing physics at non-zero temperature. Quantities within this framework is understood in terms of statistical ensembles. Among the most common statistical ensembles are the microcanonical, the canonical, and the grand canonical ensemble. In this section the two first is briefly introduced in order to discuss the relation between the canonical ensemble and the path integral approach towards quantum gravity. The appendix is written in units such that $G_{\mathrm{N}} = c = \hbar = k = 1$.

The microcanonical ensemble consists of a collection of copies of the same system one for each state accessible at a particular energy E. Each system is thought of being an isolated box with fixed energy E. Let the number of possible microstates of a given system with energy E be denoted $\Omega(E)$ then the entropy is defined to be

$$S = k \ln \Omega(E) \quad .$$

Note Boltzmann's constant is included explicit here. The entropy in the microcanonical ensemble plays the role of defining all the thermodynamics of the system. For example given the entropy as a function of the energy it is possible to adopt the thermodynamic definition of temperature

$$\frac{1}{T} = \left.\frac{\partial S}{\partial E}\right|_V \quad .$$

In the canonical ensemble the system is allowed to exchange energy with a heat bath characterized by a temperature T. Let the system be constituted of a collection of microstates with energy E_n. Then based on the assumptions about the system it is possible to assign a consistent probability to each state of energy E_n. That is the sum over all probabilities is properly normalised

$$P_n = \frac{1}{Z} e^{-\beta E_n} \quad .$$

This is known as the Boltzmann distribution. Based on the normalisation condition and the probability distribution the partition function is

$$Z = \sum_n \exp(-\beta E_n) \quad \text{where} \quad \beta = \frac{1}{T} \quad .$$

This function is essential for computing the thermodynamical quantities in the ensemble. From the probability or equivalently the partition function, the average energy can be determined by summing over all state energies weighting with the corresponding probability

$$\langle E \rangle = \sum_n P_n E_n = \frac{1}{Z} \sum_n E_n e^{-\beta E_n} = -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \quad .$$

The Holmholtz free energy is

$$F \equiv \langle E \rangle - TS = -T \ln Z \quad .$$

Like the average energy, the entropy can be computed in much the same way

$$S = -\sum_n P_n \ln P_n = -\frac{1}{Z} \sum_n e^{-\beta E_n} \left(-\beta E_n - \ln Z \right) = \beta \langle E \rangle + \ln Z \quad .$$

Finally, the heat capacity at a constant volume is given by

$$\begin{aligned} C_{\mathrm{V}} &= \frac{\partial \langle E \rangle}{\partial T} = \frac{d\beta}{dT} \frac{\partial \langle E \rangle}{\partial \beta} = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta} \\ &= \beta^2 \left[\frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \frac{1}{Z^2} \frac{\partial Z}{\partial \beta} \right] \quad . \end{aligned}$$

C.1 The Path Integral

In the following, the relation between the Euclidean path integral and the ordinary thermodynamic partition function of the canonical ensemble is established. As in the quantum field theory one considers in the case of quantum gravity path integrals over configuration space of the fields

$$Z = \int \mathcal{D}[g, \phi] e^{iI[g,\phi]} \quad . \tag{79}$$

Though, here the measure $\mathcal{D}[g,\phi]$ is on the space of metrics g and on the space of matter fields ϕ, while the action is denoted by $I[g,\phi]$. An important thing to note is that the integral is taken over both metrics that can be continuously deformed into flat space metrics, but also metrics containing singularities such as black hole metrics [9].

Evaluation of the action has several problems. First, for Lorentzian metrics and real matter fields the action is real, the path integral is therefore oscillatory and will suffer under convergence problems. Second, the action of a black hole spacetime metric must necessarily contain a singularity which makes the evaluation troublesome. Both of these complications can to some degree be redeemed by performing a clockwise rotation of the time coordinate in the complex plane such that $t = -i\tau$. This leads to the Euclidean action $I_E = -iI$ which is then evaluated on a section through the complexified spacetime, known as the the Euclidean section. On this section the metric has positive-definite signature. The consequence of this operation is that the path integral is made exponentially damped and the singularities present on the Lorentzian section may not be present on the positive-definite section.

Similar to the path integrals of quantum field theory the amplitude from an initial configuration (g_1,ϕ_1) at time t_1 to a final configuration (g_2,ϕ_2) at time t_2 in quantum gravity is given by

$$\langle (g_2,\phi_2),t_2 \mid (g_1,\phi_1),t_1\rangle = \int \mathcal{D}[g,\phi] e^{iI[g,\phi]} \quad .$$

This amplitude is also given in the Schrdinger picture using the time evolution operator e^{-itH} such that

$$\langle (g_2,\psi_2) \mid e^{-iH(t_2-t_1)} \mid (g_1,\psi_1)\rangle \quad ,$$

where H is the Hamiltonian of the system. To make contact to the canonical ensemble one is particular interested in the case of fixed initial and final configuration, say (g,ϕ). Choosing the period $t_2 - t_1 = -i\beta$ and summing over the complete set of eigenstates of the Hamiltonian one identifies the partition function for the canonical ensemble consisting of the configuration (g, ϕ) at temperature $T = \beta^{-1}$

$$Z = \sum_n \exp(-\beta E_n) \quad ,$$

where E_n is the energy of the n'th eigenstate. The trace corresponds to imposing a periodicity β in imaginary time τ. The path integral defining the partition function is thus to be taken over fields that are periodic in τ with periodicity β.

With this identification it is now possible to extract thermodynamical quantities. This is done in a semi-classical limit, using the saddle-point approximation, where it is assumed that the most dominant configurations of the path integral are controlled by metrics and matter fields which are near an extremum of the action and have the correct periodicity. If the path integral is simply evaluated at these extremum points one obtains the contribution of the background fields to the partition function

$$\begin{aligned} Z &= e^{-I_E} = e^{-\beta F} \\ \ln Z &= -I_E = -FT^{-1} \quad , \end{aligned}$$

where F is the thermodynamic potential given in equation C. Quantities such as the average energy and the entropy can be calculated as previously

$$\begin{aligned} \langle E \rangle &= -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial I_E}{\partial \beta} \\ S &= \beta \langle E \rangle + \ln Z = \beta \langle E \rangle - I_E \quad . \end{aligned}$$

It is in order to mentioned that the path integral approach also allows to compute one-loop corrections to the partition function. These corrections account for fluctuations around the background metric. Though, these terms will not be considered.

From the above it is deduced that the temperature of a given metric solution is determined by identifying the period β of the imaginary time coordinate τ. The first step is to analytic continuation the metric to Euclidean time by $\tau = it$. The periodicity of the τ must then be chosen exactly to make the Euclidean section of the metric regular at the horizon, thus removing the conical singularity at the horizon (see [10], [17]). The additional work of obtaining the corresponding action enables access to the macroscopic thermodynamical quantities of the canonical ensemble.

In the canonical ensemble the system is coupled to an infinite reservoir at constant temperature. It should be pointed out that even though a black hole in asymptotic flat space can be in equilibrium with thermal radiation at a constant temperature this configuration is unstable. This is manifest in the negativeness of the specific heat. One can understand this by letting the black hole increase its mass infinitesimally thereby getting cooler increasing its rate at which it is absorbing. The canonical ensemble is therefore unstable and one must turn to the microcanonical description to get sensible results [10]. However, as pointed out in [14] the microcanonical ensemble is somehow unphysical because the need of an insulating box that also prevent gravitons from escaping. As explored in section 5, the situation is different for a black hole in anti-de Sitter space and one can actually use the canonical ensemble.

D The Theory of Black Holes

The classical theory of black holes in four dimensions is well-established under some significant physical assumptions, such as spacetime being stationary and asymptotically flat. It is the purpose of this section to highlight some of the fundamental theorems of black hole solutions and to identify the relationship between black hole quantities and the four laws of thermodynamics. In particular, the assumptions which they are based on. Equipped with this knowledge one can generalise the result to black holes in an anti-de Sitter background. For convenience the section is written in units $G_{\rm N} = c = \hbar = k = 1$.

D.1 Uniqueness Theorems

The non-linearity of Einstein's equations makes it difficult to find exact solutions. Typically exact solutions are obtained by restricting attention to the vacuum equations $\mathcal{R}_{\mu\nu} = 0$ simplifying the complexity considerably. Another ingredient is to assume that the solution possesses symmetry (see section D.6). One such symmetry is the spherical symmetry which leads to the theorem of Birkhoff [5].

Theorem 1 (Birkhoff's theorem) *A spherical symmetric spacetime which solves the vacuum Einstein equation $\mathcal{R}_{\mu\nu} = 0$ is static.*

This implies that the only unique solution is that of Schwarzschild. Since a spacetime being static does not imply that it is spherically symmetric Birkhoff's theorem can not be used if only the exterior metric is know to be static. However, Israel's theorem can be of use if the spacetime is actually a black hole solution with an event horizon [26].

Theorem 2 (Israel's theorem) *An asymptotically flat, static, vacuum spacetime which is non-singular on and outside the event horizon is the Schwarzschild solution.*

A spacetime that is stationary is either non-rotating or it possesses axisymmetry as well, but note that the associated Killing vector fields does not necessarily commute. A third theorem is due to a number of people who established the foundation for what is known as the no-hair theorem [16].

Theorem 3 *If a spacetime is a stationary asymptotically flat vacuum black hole solution which is non-singular on and outside the event horizon then it is a two-parameter Kerr spacetime characterized by only its mass and charge.*

Finally, the uniqueness can be generalised to the Einstein-Maxwell equations. Hence a stationary asymptotically flat electrovac black hole spacetime solution must be of the Kerr-Newman black hole family and is characterised by three-parameters: the mass, the charge, and the angular momentum. This family will be introduced in the next section.

D.2 Classical Black Hole Solutions

It is expected that the final state of a gravitational collapse will be a stationary, electrovac black hole. It is therefore fortunate that the uniqueness theorems can be generalized to the vacuum Einstein-Maxwell equations for which the result is that all stationary asymptotically flat black hole spacetimes can be characterized by three parameters - the mass M, the charge e, and the angular momentum $a = J/M$. These solutions are known as the 3-parameter Kerr-Newman family also referred to as the charged Kerr family. The family spans all the classical black hole metrics for a suitable choice of parameters.

- If $e = 0$, the solution reduces to the Kerr vacuum solution.
- If $a = 0$, the solution reduces to the Reissner-Nordstrom electrovac solution.

- If $a = e = 0$, the solution reduces to the Schwarzschild solution.

The topology of a black hole is always $\mathbf{S}^2$ in four dimensions. There are several different ways of choosing coordinates for the Kerr-Newman metric (see e.g. [27], [8]). In the following the Boyer-Lindquist coordinates of the Kerr-Newman black hole will be used. The location of the event horizons of the charged Kerr metric is given by a quadratic equation with solutions [27]

$$r_{\pm} = M \pm (M^2 - a^2 - e^2)^{\frac{1}{2}} \quad . \tag{80}$$

This leads to a classification by the determinant. When $e^2 + a^2 > M^2$ the metric will have a naked singularity and the charged Kerr solution fails to be strongly asymptotically predictable, therefore it does not describe a black hole. In the case of positive determinant $e^2 + a^2 < M^2$, there exist two real event horizons. At the bifurcation point $e^2 + a^2 = M^2$ the solution is referred to as extremal. In this limit the two event horizons are coincident. Naturally, for Reissner-Nordstrom black holes where $a = 0$ solutions are called extremal if they satisfy the condition $M^2 = e^2$.

Observers in the ergosphere of the charged Kerr black hole will inevitable rotate with an angular velocity. In the limit of the event horizon $\mathcal{H}$ given by radial coordinate r_+ this takes the form [27]

$$\Omega_{\mathcal{H}} = \frac{a}{r_+^2 + a^2} \quad . \tag{81}$$

D.3 Black Hole Properties

In order to progress the investigation for locating the underlying assumptions for establishing the black hole thermodynamics a number of properties and notions for black hole solutions should be introduced.

The notion of a Killing horizon plays a fundamental role for the zeroth and first law of black hole thermodynamics. The reason for this will be elaborated in section (D.4). If there exist a Killing vector field ξ^μ which becomes normal to some null hypersurface $\mathcal{K}$ then $\mathcal{K}$ is a Killing horizon. It is of interest to know when an event horizon of a black hole is in fact a Killing horizon. There are two important results relying on the static, stationary, and axisymmetric symmetries. The first shows that in the case of a static black hole solution an event horizon must be a Killing horizon [28]. Furthermore, in a stationary and axisymmetric black hole solution, where the associated Killing vector fields ξ^μ and ψ^μ commute in the sense of section (D.6), the linear combination

$$\chi^\mu = \xi^\mu + \Omega_{\mathcal{H}}\psi^\mu \tag{82}$$

is a Killing vector field and is normal to the event horizon. Both conclusions are based on purely geometrical reasons. The second result proves that the event horizon of any stationary black hole must be a Killing horizon if it satisfies the vacuum or electrovac field equations [13]. Thus, the spacetime is either static or axisymmetric, but does not necessarily satisfy the $\xi - \phi$ orthogonality. The results are referred to as rigidity theorems.

D.4 Black Hole Thermodynamics

The quantities related to the analogy between black holes and thermodynamics is explored. It will be assumed that the appropriate parameters already have been identified as the correct thermodynamic quantities associated with a thermal object. In particular that mass and energy is equivalent and that the surface gravity and the area of the event horizon are the actual thermodynamic temperature and entropy, respectively. In the following, the event horizon is assumed to be a Killing horizon and will therefore be referred to simply as the horizon.

It is perhaps suitable to start by introducing the quantity κ, denoted the surface gravity. The name originates from its purpose in static spacetimes where it is the acceleration of an static observer near the black hole horizon measured at spatial infinity, but it is referred to as the surface gravity in much more general cases. The consequence of a Killing vector field χ^μ being normal to the horizon $\mathcal{H}$ means that the inner product $\chi^\mu\chi_\mu$ is constant on the horizon. It therefore satisfies the geodesic equation on the horizon. The surface gravity κ is often introduced as an arbitrary function on the geodesic curve needed to account for the non-affine parametrized integral curves

$$\begin{aligned} \nabla^\nu(\chi^\mu\chi_\mu) &= -2\kappa\chi^\nu \\ \chi^\mu\nabla_\mu\chi^\nu &= -\kappa\chi^\nu \quad . \end{aligned} \tag{83}$$

To every Killing horizon the surface gravity therefore exists. The orthogonality of the Killing vector field to the horizon validates Frobenius's theorem $\chi_{[\alpha}\nabla_\beta\chi_{\gamma]} = 0$ (which also is used in order to define a static metric). Utilising this together with Killing's equation $\nabla_\beta\chi_\gamma = -\nabla_\gamma\chi_\beta$ gives

$$\chi_{[\alpha}\nabla_\beta\chi_{\gamma]} = 2\left[\chi_\alpha\nabla_\beta\chi_\gamma - \chi_\beta\nabla_\alpha\chi_\gamma + \chi_\gamma\nabla_\alpha\chi_\beta\right] = 0 \quad ,$$

thus,

$$\chi_\gamma\nabla_\alpha\chi_\beta = -2\chi_{[\alpha}\nabla_{\beta]}\chi_\gamma \quad . \tag{84}$$

Contracting this equation with $\nabla^\alpha\chi^\beta$ and using the symmetry of Killing's equation

$$\begin{aligned} \chi_\gamma(\nabla^\alpha\chi^\beta)(\nabla_\alpha\chi_\beta) &= -2(\nabla^\alpha\chi^\beta)(\chi_{[\alpha}\nabla_{\beta]}\chi_\gamma) \\ &= -2(\chi_\alpha\nabla^\alpha\chi^\beta)\nabla_\beta\chi_\gamma \\ &= -2\kappa\chi^\beta\nabla_\beta\chi_\gamma \\ &= -2\kappa^2\chi_\gamma \quad . \end{aligned}$$

Finally, using that this holds for every χ_γ on the horizon an explicit formula for evaluation of the surface gravity has appeared

$$\kappa^2 = -\frac{1}{2}(\nabla^\alpha\chi^\beta)(\nabla_\alpha\chi_\beta)\Big|_{\mathcal{H}} \quad . \tag{85}$$

Where the evaluation on the horizon is stated explicitly. Defining the covariant derivative with respect to the Kerr-Newman metric and using the Killing vector field given by equation (82) the surface gravity of the horizon of an asymptotically flat stationary black hole solution can be computed

$$\kappa = \frac{(M^2 - a^2 - e^2)^{\frac{1}{2}}}{2M[M + (M^2 - a^2 - e^2)^{\frac{1}{2}}] - e^2} \quad . \tag{86}$$

where M, a, e are the parameters defined in section (D.2). Setting $a = e = 0$ this reduces to the result of the static Schwarzschild metric

$$\kappa = \frac{1}{4M} \quad .$$

This is the acceleration of a static observer located near the horizon as measured by a static observer at infinity. It is important to note that the surface gravity is not a property of the Killing horizon alone, but it depends also on the normalisation of χ^μ. For asymptotically flat spacetimes one chooses the normalisation at spatial infinity

$$\chi^\mu \chi_\mu \to -1 \quad \text{as} \quad r \to \infty \quad .$$

The redshift factor is the magnitude of the Killing vector $V = (-\chi_\mu \chi^\mu)^{\frac{1}{2}}$. Thus with the choice of normalisation, the redshift factor at infinity is one and zero at the horizon. The redshift factor relates emitted and observed energies of a photon as measured by a static observer. Denote the energy of a photon E measured at a point and the energy measured by an observer at infinity by E_∞ then

$$E = \frac{E_\infty}{V} \quad . \tag{87}$$

To establish the zeroth thermodynamic law of black holes the surface gravity is required to be constant over the horizon. Having a surface gravity defined as equation (85) required the crucial property of the event horizon being a Killing horizon. From this alone the geodesic equation (83) implies that κ is constant along the orbits of the Killing vector field by taking the Lie derivative

$$\mathcal{L}_\chi \kappa = 0 \quad . \tag{88}$$

The two rigid theorems given in section D.3 leads to two different ways of extending this to the entire horizon. First, on purely geometrical assumptions if the spacetime is static or stationary, axisymmetric with the $\xi - \phi$ orthogonality property satisfied. Alternatively, it can be shown by assuming Einstein's equations with the dominant energy condition satisfied, which means the $\xi - \phi$ orthogonality condition is relaxed. In either case the surface gravity will be constant over the entire event horizon

$$\chi_{[\mu} \nabla_{\nu]} \kappa = 0 \quad .$$

The identification between the temperature and surface gravity $T \leftrightarrow \alpha\kappa$ with α being a constant of proportionality is that a body in thermal equilibrium have constant temperature throughout the body is equivalent to the surface gravity being constant over the entire horizon. More generally solutions which satisfy the zeroth law can be divided into two cases. The non-extremal black holes with $\kappa \neq 0$ for which one can show that a Killing horizon with a bifurcation two-surface imply that κ is constant and the case of extremal black holes with $\kappa = 0$ where one will have to use the dominant energy condition in order to ensure constant κ.

The notion of energy in general relativity is made more subtle because in addition to the energy content of matter one also have to include the gravitational field energy. Although, a local energy density of the matter content makes sense due to the conservation of the stress-energy tensor, it is near impossible to express the energy density of the gravitational field. One must therefore discard this notion of a local energy density. Instead, one considers the notion of a total energy of an isolated system. An isolated system in general relativity can be an asymptotically flat spacetime. First, the energy, charge, and angular momentum shall be considered in the case of stationary asymptotically flat spacetimes where the Komar integral can be utilised. Thereafter, the total energy for non-stationary asymptotically flat spacetimes at null and spatial infinity known as the Bondi mass and ADM energy, respectively, will be considered. Finally, the positive energy theorem for the Bondi and ADM energy is discussed.

First, the Komar integral shall be introduced. Let Σ be some spacelike hypersurface and $\partial\Sigma$ its boundary such that $\Sigma \cup \partial\Sigma$ is a closed compact manifold with boundary then for every Killing vector field ξ^μ there exist a conserved charge Q given by the Komar integral

$$Q = \frac{k}{16\pi} \oint_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} n_\mu \sigma_\nu \nabla^\mu \xi^\nu \quad , \tag{89}$$

which by Stokes's theorem can be written

$$Q = \frac{k}{16\pi} \int_{\Sigma} d^3x \sqrt{\gamma^{(3)}} n_\mu \nabla_\nu \nabla^\mu \xi^\nu \quad . \tag{90}$$

Here k is some constant, $\sqrt{\gamma^{(3)}} d^3x$ and $\sqrt{\gamma^{(2)}} d^2x$ constitute the volume element on Σ and $\partial\Sigma$, respectively. The unit normal vector to Σ is denoted n_μ and the outpointing normal to the boundary $\partial\Sigma$ is denoted σ_ν. It is common to choose Σ to be the interior of a spacelike 2-surface (a topological 2-sphere) lying on the hypersurface orthogonal to ξ^ν. The boundary $\partial\Sigma$ can then be thought of to enclose all sources. The value of the integral is independent of the choice of 2-surface Σ due to ξ^ν being a Killing vector field.

Now, as mentioned in section (D.6) any Killing vector field ξ^μ satisfies equation (101). Thus, one can write the integrand as

$$\begin{aligned} J^\mu &= \nabla_\nu \nabla^\mu \xi^\nu \\ &= \xi_\mu R^{\mu\nu} \\ &= 8\pi \xi_\mu \left[T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right] \quad , \end{aligned} \tag{91}$$

where the last equality is obtained using Einstein's equations. T denote the trace of $T^{\mu\nu}$. Taking the divergence and using the contracted Bianchi identity $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$, the symmetry of Killing's equation, and equation (100) this expression turns out to be divergenceless. Hence, the charge given by the Komar integral (89) is independent of time and the integrand J^μ is a conserved current.

Specifically, for the Kerr-Newman solutions which are cases of stationary and axisymmetric asymptotically flat spacetimes, the Komar integral can be used to define the conserved quantities: total energy/mass, the angular momentum, and in a much similar fashion the electromagnetic charge.

In the Komar integral the Ricci tensor is utilised to construct the current. In stationary spacetimes an exact time translation symmetry exist generated by the timelike Killing vector field ξ^μ. Using formula (89) directly the total energy at spatial infinity can be written

$$E = -\frac{1}{8\pi}\oint_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu\xi^\nu \quad , \tag{92}$$

where a constant of $k = -2$ has been chosen for convenience. The Komar integral approach also works for axisymmetric spacetimes. Here the existence of a rotational Killing vector field ψ^μ is guaranteed. Thus, a conserved current of the form of equation (91) exists and equation (89) gives the total angular momentum

$$J = \frac{1}{16\pi}\oint_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu\psi^\nu \quad . \tag{93}$$

Note that the coefficient is chosen differently $k = 1$. Finally, for the electric charge the conserved current is given directly from Maxwell's equations $J^\mu = \nabla_\nu F^{\mu\nu}$, where $F^{\mu\nu}$ is the field strength tensor. The charge passing through a spacelike hypersurface Σ can likewise be written as an integral over the boundary using Stokes's theorem

$$Q = -\oint_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu F^{\mu\nu} \quad . \tag{94}$$

Note the convention of sign. Interestingly, the integral shows that the only thing needed to obtain the total charge of a spacetime is the behaviour of the electromagnetic field at spatial infinity. using the dual field strength tensor the magnetic charge could have been obtained in a similar fashion.

Non-stationary asymptotically flat spacetimes does not possess exact time translation symmetry like the stationary case, but instead they have an asymptotically timelike Killing vector field ξ^μ. Roughly, this implies that as one tend to null infinity, the vector $(\partial_t)^\mu$, will become a better and better approximation to Killing's equation and one can exploit the Komar integral formulation. This defines the Bondi energy which accounts for the loss of energy due to gravitational radiation under a gravitational collapse. In the null infinite limit it will take the same form as the Komar integral given by equation (92) over the 2-sphere.

The total energy for spatial infinity can be defined using that the metric tends to Minkowski flat $\eta_{\mu\nu}$ at infinity. One define the differences from the flat metric as

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \quad , \tag{95}$$

so $h_{\mu\nu}$ will tend to zero at spatial infinity. The ADM energy can then be derived to be

$$E_{\text{ADM}} = -\frac{1}{16\pi} \oint_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} \sigma^j \left(\frac{\partial h_{ij}}{\partial x^i} - \frac{\partial h_{ii}}{\partial x^j} \right) \quad , \tag{96}$$

where i, j are the spatial indices. One can show that the ADM energy agrees with the Komar energy in the stationary case.

Definitions of total energy must be positive for physical configuration. The proof of the positiveness of the energy was given by Shoen and Yau and later in an alternative form by Witten. It essentially states that an asymptotically flat spacetime satisfying Einstein's equations with the dominant energy condition and the existence of a non-singular Cauchy surface will have non-negative ADM energy. It also follows that the only spacetime that have zero energy is the Minkowski spacetime. Singular spacetimes such as Schwarzschild with positive mass do not have negative energy, but in order to include singular spacetimes they must have evolved from a non-singular initial data set.

All the ingredients to write down the first law of black hole thermodynamics is now given [3]. Including the work from the electric potential Φ on the horizon the differential form for the mass is

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_{\mathcal{H}}\delta J + \Phi_{\mathcal{H}}\delta Q \quad . \tag{97}$$

For strongly asymptotically predictable spacetimes with matter satisfying the weak or strong energy conditions, the area theorem states that the area of the event horizon never decreases $\delta A \geq 0$. This is a purely classical result along with the fact that black holes have absolute zero temperature. This, in particular, would imply an inconsistency in the physical relationship between S and A. However, the consequences from quantum mechanical effects alter the picture.

Although, no complete quantum gravity theory exist, the significant result that black holes emit particles due to quantum mechanical effects has been shown [12]. To deduce this, the matter fields is assumed to obey the same wave equations with the Minkowski metric replaced by the classical curved spacetime metric (minimal coupling). This metric satisfies the Einstein's equations where an expectation value of the energy-momentum tensor is used. Through a series of assumptions, it was shown that a black hole radiates a total number of particles proportional to Fermi-Dirac/Bose-Einstein statistics depending on whether the particles are fermions or bosons. For particles with non-zero rest mass the argument differs slightly, but the result is the same. A black hole therefore radiates with a temperature T exactly as a black body. One identifies

$$T = \frac{\kappa}{2\pi} \quad .$$

The quantum mechanical treatment have the important consequence that the expectation value of the energy-momentum tensor violates the energy conditions assumed in the area theorem. Knowing the exact relation between temperature and surface gravity the corresponding relation between entropy and area can be deduced from the first law (97)

$$S = \frac{A}{4} \quad . \tag{98}$$

The quantum effects imply that the total entropy of the system should be viewed as the total entropy S of the matter outside the black hole plus the entropy of the black hole. One is in this way lead to writing the second law of thermodynamics as

$$\delta S' \geq S + \frac{A}{4} \quad , \tag{99}$$

which is known as the generalised second law (GSL).

Finally, from equation (86) it is seen that κ only vanishes for the extremal case $M^2 = a^2 + e^2$. One can show that the more extreme the black hole is, the harder it is to get even closer to the extremal condition. This is consistent with the third law of thermodynamics.

D.5 The Chain of Assumptions

The classical black hole solutions in four dimensions are established on stationary, asymptotically flat spacetimes. In defining surface gravity it was crucial that a Killing vector field which was normal to the event horizon existed. Note, that static spacetimes is guaranteed to have such a Killing vector field which makes a surface gravity for black holes in anti-de Sitter space possible. The zeroth law was based on the rigid theorems. These are based on the assumption that the spacetime either has a specific symmetry or that the matter satisfied dominant energy conditions. The area theorem assumes the spacetime is strongly asymptotically predictable as well as the matter satisfying the weak or strong energy conditions. Finally, a discussion of conserved quantities, especially the energy for which the positive mass theorems are relevant, was done. It was shown that a suitable definition for the total energy of a non-stationary spacetime is given by the ADM formula.

Including the cosmological constant in Einstein's equations implies that the otherwise vacuum solutions necessarily will have curvature as explored in detail in appendix A. This naturally raises a number of questions in the light of the assumptions made to establish the thermodynamic relationship. Especially, it should be mentioned that the considered spacetimes in this section have been global hyperbolic, that is have a Cauchy surface and therefore have a well-defined initial value problem. This is not the case for the negative curved anti-de Sitter space, since infinity is timelike, however, introducing boundary conditions it is still possible to put physics in anti-de Sitter. This will be elaborated in more detail in section 4.2.

D.6 Notion of Symmetry in General Relativity

The intrinsic curvature of a spacetime $(M, g_{\mu\nu})$ makes the notion of symmetry a more difficult concept in general relativity than in other fields of physics. However, without properties like them, it would be very difficult to obtain and progress upon solutions to Einstein's non-linear equations. This section will contain a brief introduction to the notion of symmetry.

Let M be a manifold. A diffeomorphism is a differentiable map $\phi : M \to M$ for which the inverse ϕ^{-1} exist and is differentiable. For a diffeomorphism, both the pullback and the pushforward of a tensor can be defined. This gives rise to a way of comparing tensors at a point, namely the Lie derivative. However, to define the Lie derivative a one-parameter group of diffeomorphisms is necessary. This is a differentiable map $\phi_t : \mathbb{R} \times M \to M$ such that for each fixed $t \in \mathbb{R}$, ϕ_t is a diffeomorphism $\phi_t : M \to M$ and for all $t, s \in \mathbb{R}$ the diffeomorphisms satisfy $\phi_t \circ \phi_s = \phi_{t+s}$. For a fixed $p \in M$ there exist a curve $\phi_t(p) : \mathbb{R} \to M$ called the orbit of ϕ_t parametrized by $t \in \mathbb{R}$. Introducing a vector field v^μ on M those integral curves x^μ parametrized by $t \in \mathbb{R}$ satisfies,

$$\frac{dx^\mu(t)}{dt} = v^\mu(x) \quad ,$$

it is possible to identify each point $p \in M$ with $\phi_t(p)$, such that, it is the point on the integral curve corresponding to parameter t. Hence, v^μ defines a one-parameter group of diffeomorphisms ϕ_t. The vector $v^\mu(p)$ will be the tangent to the orbit $\phi_t(p)$ evaluated at $t = 0$ and is said to be the generator of the diffeomorphism.

It is now possible to pull back the value of a tensor at point $\phi_t(p)$ and compare it to its value at p. Thus, the Lie derivative measures how a tensor is changing along the integral curves. It is defined as a map $\mathcal{L}_V : (k, l) \to (k, l)$ which is linear and obeys the Liebniz rule. As a special case of the Lie derivative it is worth mentioning the Lie bracket which is defined as

$$\mathcal{L}_v u^\mu = [v, u]^\mu = v^\lambda \partial_\lambda u^\mu - u^\lambda \partial_\lambda v^\mu \quad ,$$

which implies $\mathcal{L}_v u^\mu = -\mathcal{L}_u v^\mu$.

Now, it is possible to introduce the notion of a symmetry. A diffeomorphism ϕ is called a symmetry of a tensor T if it leaves the pullback invariant, $\phi^* T = T$. Similar if one has a one-parameter group of diffeomorphisms ϕ_t generated by a vector field v^μ it is said to be a symmetry of T if it satisfies

$$\mathcal{L}_v T = 0 \quad .$$

For a spacetime $(M, g_{\mu\nu})$ one can have a symmetry of the metric $\phi^* g_{\mu\nu} = g_{\mu\nu}$. Such a diffeomorphism is referred to as an isometry. If one has a one-parameter family of isometries generated by a vector field ξ^μ then ξ^μ is called a Killing vector field

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \quad .$$

This can be rewritten by evaluating the Lie derivative of the metric with respect to an arbitrary vector field v^μ,

$$\mathcal{L}_v g_{\mu\nu} = 2\nabla_{(\mu} v_{\nu)} \quad ,$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$. The above equation is then Killing's equation

$$\nabla_{(\mu} \xi_{\nu)} = 0 \quad .$$

Analogous to the above, the Killing vector field is said to generate the isometry and a symmetry is a property of the metric which leaves it unchanged along the direction of the Killing vector field. In that sense, a symmetry characterize the underlying geometry of the manifold. Another way of stating that the geometry is unchanged is to consider the directional derivative of the Ricci scalar which vanishes along a Killing vector field

$$\xi^\lambda \nabla_\lambda \mathcal{R} = 0 \quad . \tag{100}$$

Any Killing vector field ξ^μ has a relation to the Riemann tensor given by, (see appendix C in [27] for proof),

$$\nabla_\mu \nabla_\sigma \xi^\rho = \mathcal{R}^\rho{}_{\sigma\mu\nu} \xi^\nu \quad .$$

Contracting over ρ and μ yields

$$\nabla_\mu \nabla_\sigma \xi^\mu = \mathcal{R}_{\sigma\nu} \xi^\nu \quad . \tag{101}$$

Taking the divergence of this equation ∇^σ, using the Bianchi identity in the form $\nabla_\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R$, and the symmetry of Killing's equation one obtains equation (100).

Finding all Killing vectors of a metric can be tricky, since they are not related in a simple way to symmetries of the spacetime, however one fact is that if the metric in question is independent of a specific coordinate x^σ, where σ is fixed, the vector $(\partial_\sigma)^\mu$ will satisfy Killing's equation.

The importance of Killing vector fields are heavily based on the fact that they imply the existence of conserved quantities in the sense of geodesic motion. Given a geodesic with tangent vector u^μ, the quantity $\xi_\mu u^\mu$ is conserved along the geodesic due to Killing's equation

$$\xi^\lambda \nabla_\lambda (\xi_\mu u^\mu) = 0 \quad . \tag{102}$$

Furthermore, one can obtain conserved currents, which is done in section (D.4) in relation to quantities associated with black holes. It is in place to introduce some notation for spacetimes possessing special symmetries.

If there exist a one-parameter group of isometries ϕ_t (which leaves the metric invariant) whose orbits are timelike curves, the associated spacetime is said to be **stationary**. Therefore, if the metric possesses a timelike Killing vector field ξ^μ whose integral curves are complete then the spacetime is stationary. Consequently, the metric is stationary if it possesses a Killing vector that is asymptotically timelike near infinity. The Killing vector field ξ^μ is said to generate time-translations $t \to t + \text{const.}$. Furthermore, if the metric is stationary and the timelike Killing vector field ξ^μ is orthogonal to a family of spacelike hypersurfaces it is said to be **static**. Mathematically, this can be written as a requirement on ξ^μ

$$\xi_{[\alpha} \nabla_\beta \xi_{\gamma]} = 0 \quad . \tag{103}$$

The condition is also the necessary and sufficient condition for the existence of time reflection symmetry $t \to -t$, that is every static spacetime has time reflection symmetry. A third symmetry, a spacetime can be said to possess, is the **axisymmetry**. A metric is said to be axisymmetric if there exist a one-parameter group of isometries χ_s whose orbits are closed spacelike curves i.e. they can be regarded as a map $\chi_s : S^1 \to M$. As above this implies the existence of a spacelike Killing vector field ψ^μ whose integral curves are closed. A spacetime is said to be stationary and axisymmetric if it has both symmetries and the associated one-parameter groups of isometries or equivalent their Killing vector fields commute, that is

$$[\xi^{\mu}, \psi^{\nu}] = 0 \quad . \tag{104}$$

This last requirement is sometimes referred to as the $\xi - \phi$ orthogonality property. The $\xi - \phi$ orthogonality property holds for all stationary-axisymmetric vacuum or electrovac black hole solutions.

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